Advanced Data Structures

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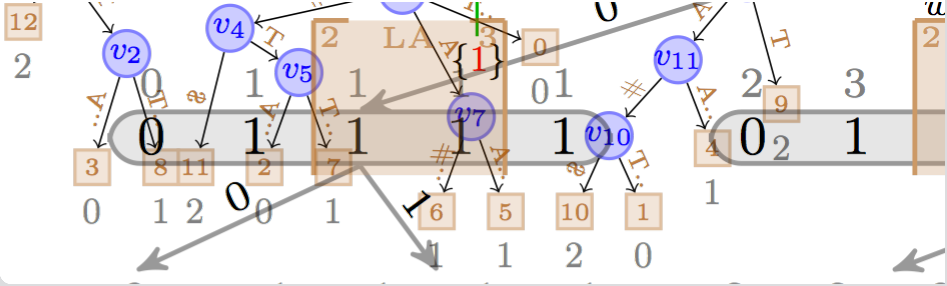


Dynamic Perfect Hashing

What we want:

- $O(1)$ lookup time (as in static perfect hashing)
- Keys not known in advance
- Good expected performance for insert

Cuckoo hashing

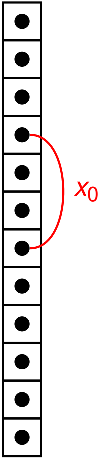
- Uses *two* hash functions h_1 and h_2
- Key x stored either at position $h_1(x)$ or at $h_2(x)$
- At most one key per position in the hash table
- Worst case lookup time: $O(1)$
- Removing a key is also constant
- Insertion of a key is $O(1)$ expected, amortized

Cuckoo hashing

lookup(x)
  i $\leftarrow h(x)$
  if $T[h_1(x)] = x$ **or** $T[h_2(x)] = x$ **then**
    return *true*
  return *false*

insert(x)
  if lookup(x) **then**
    return
  $p \leftarrow h_1(x)$
  for $i \leftarrow 0$ **to** $n - 1$ **do**
    if $T[p] = \perp$ **then**
      $T[p] \leftarrow x$; **return**
    swap(x, $T[p]$)
    $p \leftarrow h_{1+(p=h_1(x))}(x)$
  rehash(); insert(x)

Cuckoo graph



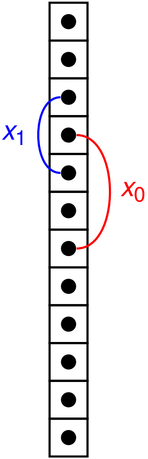
The *cuckoo (undirected) graph* consists of

- m nodes (one for each table entry)
- For each key x thre is on edge connecting $h_1(x)$ and $h_2(x)$

Example

- Including key x_5 causes a cycle. Are cycles dangerous? What is the probability of getting a cycle?
- Including key x_6 will result in a rehash

Cuckoo graph



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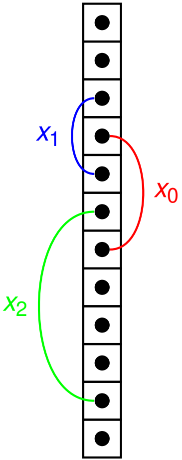

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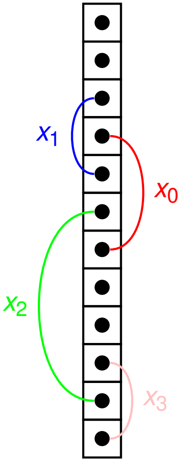

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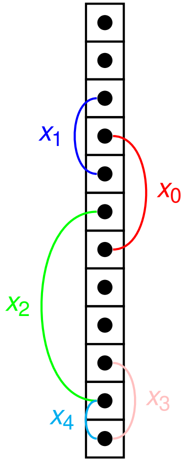

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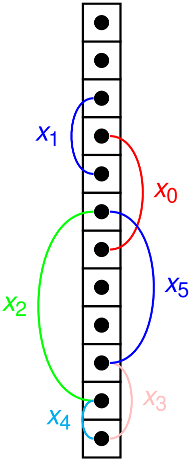

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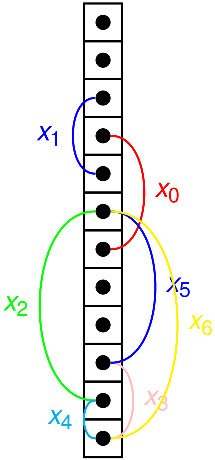

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Cuckoo hashing – analysis

Assumptions

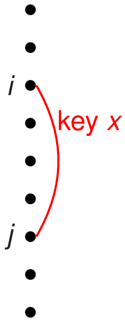
- Keys have the same size and can be compared in constant time
- Tow hash functions h_1 and h_2 which map to $[m]$. The probability for any function value $h_i(x)$ to be a particular value in $[m]$ is $\frac{1}{m}$. Function values are independent of each other.
- Fixed upper bound n on the number of keys in the set S.

Cuckoo hashing – analysis

Lemma ([1])

For any position i and j, and any $c > 1$, if $m \geq 2cn$ then the probability that in the undirected cuckoo graph there exists a path from i to j of length $\ell \geq 1$, which is a shortest path from i to j, is at most $\frac{1}{c^\ell m}$.

Proof (by induction)



- Base case: $\ell = 1$
- For each $x \in S$ we have $\Pr(x \text{ mapped to node } i \text{ and } j) = \frac{2}{m^2}$, since either $h_1(x) = i \wedge h_2(x) = j$ or $h_1(x) = j \wedge h_2(x) = i$
- Using union bound, we get that the probability that there is an edge between i and j is at most

$$\sum_{x \in S} \frac{2}{m^2} \leq \frac{2n}{m^2} \overset{n \leq \frac{m}{2c}}{\leq} \frac{1}{cm}$$

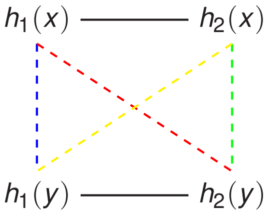
Cuckoo hashing – analysis

Proof continued

- Inductive step: $\ell > 1$ and lemma holds for length $\leq \ell - 1$
- If there is a path between i and j of length ℓ but not shorter than ℓ then there must be a position k such that
  - A there is a shortest path of length $\ell - 1$ from i to k that does not go through j, and
  - B there is an edge from k to j
- $\Pr(A) \leq \frac{1}{c^{\ell-1}m}$, by induction hypothesis and the fact the requirement „does not go through j" makes the probability even smaller
- $\Pr(B|A) = \sum_{x \in S} \frac{2}{m^2} \leq \frac{1}{cm}$
- $\Pr(A \text{ and } B) = \Pr(A) \cdot \Pr(B|A) \leq \frac{1}{c^{\ell-1}m} \cdot \frac{1}{cm} = \frac{1}{c^{\ell}m^2}$
- Sum of all possible k and using union bound gives an upper bound on the probability of a shortest path of length ℓ between i and j of $\frac{1}{c^{\ell}m}$ $\square$

Cuckoo hashing – analysis

- Two keys are in the same bucket if a path connects $\{h_1(x), h_2(x)\}$ and $\{h_1(y), h_2(y)\}$ in the cuckoo graph (there are 4 possible ways to do this)



- Probability of two keys $x \neq y$ to be in the same bucket can be upper bounded by

$$4\sum_{\ell=1}^{\infty}\frac{1}{c^{\ell}m} = \frac{4}{(c-1)m} = O\left(\frac{1}{m}\right)$$

Cuckoo hashing – analysis

- Assume there are no cycles in the cuckoo graph
- From the previous lecture we know that the time for an operation is bounded by the number of elements in the bucket
- With the same analysis we get that the expected time per operation is $O(1)$ and $O(1)$ worst case on lookups (Assuming $m \geq 2cn$).

Next, analysis of the cost of rehashing...

Cuckoo hashing – analysis

Rehashing

- Consider sequence of operations involving ϵn insertions (e.g. $\epsilon = 0.1$)
- Let S' be the set of keys that exists at some time during insertions
- How likely is a cycle (=path from node i back to itself)? With the previous lemma we can upper bound that a position i is involved in a cycle

$$\sum_{\ell=1}^{\infty} \frac{1}{c^{\ell} m} = \frac{1}{(c-1)m}$$

- Using union bound, we get an upper bound for the probability that there is at least one cycle:

$$\sum_{i=1}^{m} \frac{1}{(c-1)m} = \frac{1}{(c-1)}$$

Cuckoo hashing – analysis

Rehashing

- For $c = 3$, the probability is at most $\frac{1}{2}$ that a cycle occurs (i.e. a rehash could be required) during the ϵn insertions
- The probability of two rehashes (caused by a second independent cycle) is $\frac{1}{4}$, and so on.
- I.e. the expected number of rehashes during ϵn insertions is at most

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

- If a rehash takes $O(n)$ time (show why?) the expected amortized time of rehashes over ϵn insertions is $O(\frac{1}{\epsilon})$, i.e. constant

Cuckoo hashing – analysis

Global rebuilding

- Adapt the size of the hash table to the number of keys.
- Whenever the set becomes too small/large compared to the size of the hash table, a new smaller/larger hash table is created.
- To guarantee constant expected amortized cost per operation the size should be decreased/increased by a constant factor.

Cuckoo hashing – analysis

- Our assumption of true randomness is not realistic.
- Original work uses concept of (c, k)-unviversal hash functions. Here the hash values of any choice of k keys are independent.
- It can be shown that cuckoo hashing still performs well using (c, k)-universal hash functions: Perform a rehash if a key cannot be inserted after $k = \log n$ steps (instead of n in true randomness case).
- Siegel [FOCS 1989] showed that $(1, O(\log n))$-universal hash functions exists (taking $O(\log n)$ space and can be evaluated in $O(1)$ time)

Bibliography

[1] Rasmus Pagh. Cuckoo hashing for undergraduates. 2006. Available online at http://www.it-c.dk/people/pagh/papers/cuckoo-undergrad.pdf.