Advanced Data Structures
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Orthogonal range searching

Classical OLAP queries: „Find all users aged between 30 and 35 which are connected to at least 100 and at most 200 other users”
Orthogonal range searching – 1D

One dimensional case ($d = 1$). Example ($x_0 = 19$, $x_1 = 76$):

- $\text{count}(x_0, x_1) = 8$
- $\text{report}(x_0, x_1) = \{22, 35, 40, 44, 54, 62, 73\}$
Simple solution

- Sort points according to $x$-coordinates ($O(n \log n)$) and store them in array $A$
- Calculate successor $x'_0$ of $x_0$ and predecessor $x'_1$ of $x_1$
- Let $i'(j')$ be the index of $x'_0$ ($x'_1$) in $A$
- Method *count* returns $k = j' - i' + 1$ (in $O(\log n)$ time)
- Method *report* returns subarray $A[i', j']$ (in $O(\log n + k)$ time)
Orthogonal range searching – 1D

Alternative solution: balanced binary search trees

- Find point in middle, split set and recurse on both half (pick left point if set size is even)
- Depth is \( \log n \), construction time is bounded by sorting \( O(n \log n) \)
Orthogonal range searching – 1D

Alternative solution: balanced binary search trees

Find successor (predecessor) of \( x_0 \) (\( x_1 \)) again
too small

too large

split node

$x_0$

$x_1$
Subtrees of off-path edges are either included or excluded from the result.
Result can be implicitly represented using included off-path subtrees (there are at most $O(\log n)$ of them).
Two dimensional case \((d = 2)\). Example \((x_0 = 12, x_1 = 32, y_0 = 10, y_1 = 29)\):

- \(\text{count}(x_0, x_1, y_0, y_1) = 11\)
- \(\text{report}(x_0, x_1, y_0, y_1) = (19, 40), (23, 39), (22, 49), \ldots\)
First attempt of a solution

- Find all points with \( x_0 \leq x \leq x_1 \)
- Find all points with \( y_0 \leq y \leq y_1 \)
- Intersect the two resulting list
- Time complexity for this approach:
  \[
  O(\log n + k_x) + O(\log n + k_y) + O(k_x + k_y) = O(\log n + k_x + k_y),
  \]
  where \( k_x \) and \( k_y \) is the length of the two lists
- Well, there are cases...
Orthogonal range searching – 2D

$k_x = 45$
$k_y = 49$
$k = 5$
$n = 91$
Orthogonal range search – 2D

Second attempt

- Build a balanced binary tree using the $x$ coordinates
- Calculate the $O(\log n)$ subtrees which contain all points with $x_0 \leq x \leq x_1$
- Idea: Filter these subtrees by $y$-coordinate
Orthogonal range searching – 2D

How to filter by $y$-coordinate

- For each node $v$ in the tree build a 1D range searching structure on the $y$-coordinates of all points in $v$’s subtree
- This can be done during the preprocessing
- How does the query process change?
  - Determine paths to successor and predecessor of $x_0$ and $x_1$
  - Determine the root nodes of the $O(\log n)$ included off-path subtrees
  - For each such root node $v_i$ retrieve all points which are in $[y_0, y_1]$ in $O(\log n + k_i)$ time, where $k_i$ is the number of matching points

Total time complexity: $O(\log^2 n + k)$

- At most $O(\log n)$ subtrees for $x$
- Retrieval time for each subtree $O(\log n + k_i)$
- Points from two different subtrees are distinct
Orthogonal range searching – 2D

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Orthogonal range searching – 2D

How much space is used?

- Tree high is $O(\log n)$
- On each level $\ell$ each point is represented in only one node
- $\sum_{i=0}^{c_1} \log n\ c_2 n = O(n \log n)$ words

How long does the preprocessing take?

- Problem: points have to be sorted according to $y$-coordinate in each node
- Solution: bottom-up construction
  - Start at the leaves
  - Merge the (already sorted) lists of the two children of a node
  - I.e. $O(n \log n)$ construction time
2d-range searching in $O(\log n + k)$ time

- Idea: Avoid expensive calculation of successor/predecessor in all $O(\log n)$ 1d-range structures for $y$-coordinates.
- Determine successor/predecessor in root node and map result into child nodes.
- Technique known as *fractional cascading*.
- More detailed: For each node $v$ and entry of the $y$-range searching structure, store a pointer to the corresponding successor in $v$’s left and right child.
Orthogonal range searching – 2D

\[ y_0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad y_1 \]

\[
\begin{array}{cccccccccccc}
6 & 11 & 17 & 32 & 42 & 46 & 51 & 60 & 69 & 80 & 94 \\
11 & 17 & 51 & 60 & 94 & 6 & 32 & 42 & 46 & 69 & 80 \\
\end{array}
\]