Karlsruhe Institute of Technology

Advanced Data Structures

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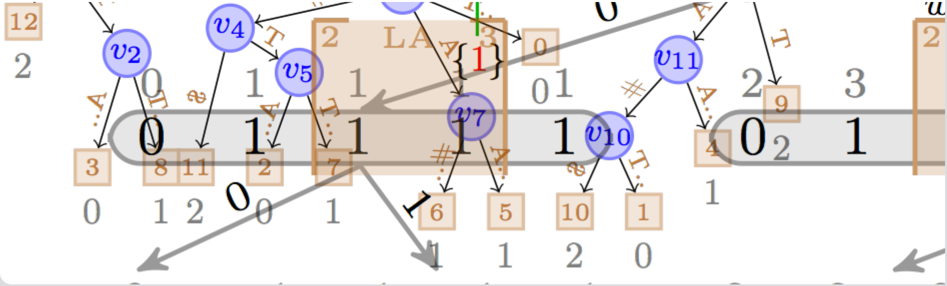

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Succinct Data Structures

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Definition

A *succinct data structure* uses space „close" to the information-theoretical lower bound, but still supports operations time-efficiently.

Let L be the information-theoretical lower bound to represent a class of objects. Then a data structure which still supports time-efficient operations is called

- *implicit*, if it takes $L + O(1)$ bits of space
- *succinct*, if it takes $L + o(L)$ bits of space
- *compact*, if it takes $O(L)$ bits of space

Example: Succinct indexable dictionary

Example 1

Represent a subset $S \subset [n]$ and support operations:

- $member(i)$ returns if $i \in S$
- $rank(i) = |\{s \mid s \in S \wedge s < i\}|$
- $select(j) = min\{k \mid rank(k+1) = j \wedge k \in [n]\}$
- *predecessor* and *successor* can be answered by using *rank* and *select*

There are 2^n different subsets, i.e. we need $\log 2^n = n$ bits of space to distinguish between dictionaries.

Example: Succinct indexable dictionary

Solution

Use a bitvector b of length n with

$$b[i] = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

plus $o(n)$-space indexes to answer

- $rank(i, 1, b) = \sum_{j=0}^{i-1} b[i]$
- $select(j, 1, b) = \min\{i \mid rank(i+1, 1, b) = j\}$

Example: Succinct indexable dictionary

$o(n)$ rank index (Jacobson, FOCS 1989)

- Partition b into super blocks of size $\alpha = \lceil \log^2 n \rceil$
- Store absolute count $A[i] = \sum_{k=0}^{\alpha \cdot i-1} b[k]$ for each block i in $\log n$ bits
- Partition each super block into blocks of size $bs = \beta = \lceil \frac{1}{2} \log n \rceil$
- Store relative count $R[i,j] = (\sum_{k=0}^{\beta j-1} b[k + \alpha i]) - A[i]$ for each block j in super block i.
- Use „four Russian trick” for blocks of size β. Pre-compute lookup table LTB of size $2^{\beta} \beta \log \beta = o(n)$
- Total space: $\frac{n \log n}{\alpha} + \frac{n \log\log n}{\beta} + O(1) = o(n)$ bits

Example: Succinct indexable dictionary

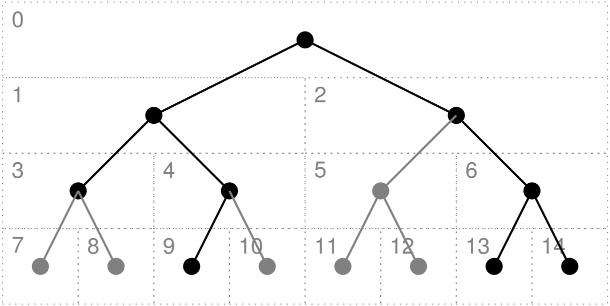
A simple solution to select: binary search over the $o(n)$-space rank structure.

$o(n)$ select structure

- See blackboard.

Succinct representation of trees

- First consider binary trees
- Number of n-node binary trees: $C_n = \frac{1}{n+1}\binom{2n}{n}$
- We need $\log C_n = 2n + o(n)$ bits (using Sterling's Approximation)
- Operations: $parent(v)$, $leftchild(v)$, $rightchild(v)$



1	1	1	1	1	0	1	0	0	1	0	0	0	1	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

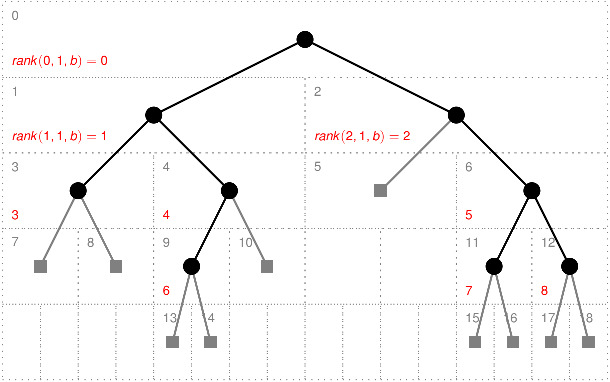
Succinct representation of trees

- In a very balanced binary tree (like in a heap) operations are easy
- Let 0 be the root identifier
  - $\mathit{parent}(v) = \lfloor \frac{v-1}{2} \rfloor$ for $v > 0$
  - $\mathit{leftchild}(v) = 2v + 1$
  - $\mathit{rightchild}(v) = 2v + 2$
- For unbalanced trees the space would be 2^d bits, where d is the maximum depth of a node.

Proposal of Jacobson (FOCS 1989):

1. Mark all the nodes of the tree with a 1.
2. Add external nodes to the tree, and mark them all with 0-bits.
3. Read off the bits marking the nodes of the tree in (left-to-right) level-order.

Succinct representation of trees



$b =$

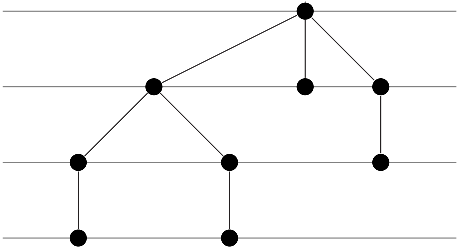
1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Succinct representation of trees

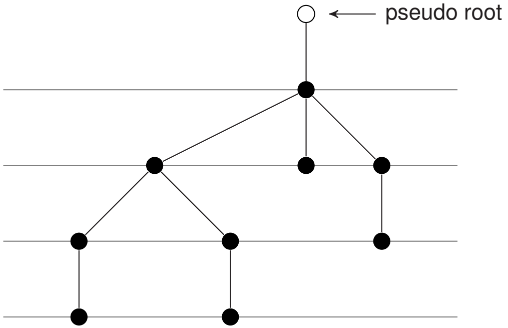
- b contains n set bits and is of length $2n+1$.
- A node is represented by the positon of its corresponing 1-bit in b.
  - $leftchild(v) = 2 \cdot rank(v) + 1$
  - $rightchild(v) = 2 \cdot rank(v) + 2$
  - $parent(v) = select(\lfloor \frac{v-1}{2} \rfloor + 1, 1, b)$ for $v > 0$
- Total space (including rank and select): $2n + o(n)$ bits

Jaboson also considered rooted, ordered tree with degree higher than 2.

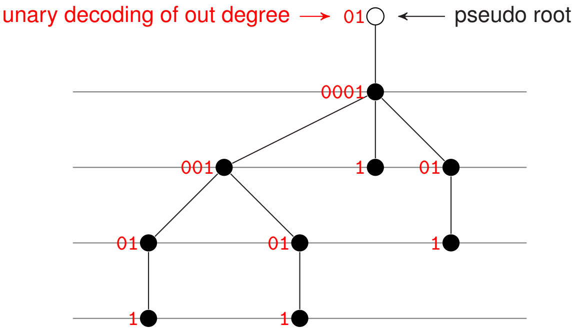
LOUDS – level order unary degree sequence



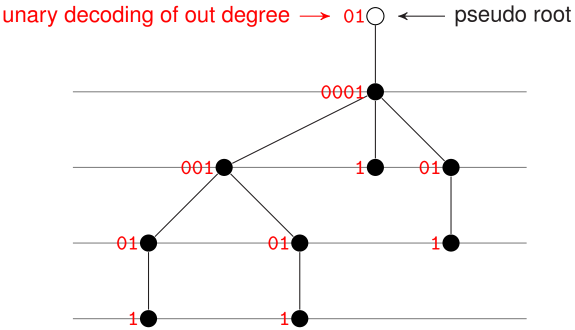
LOUDS – level order unary degree sequence



LOUDS – level order unary degree sequence



LOUDS – level order unary degree sequence



LOUDS sequence = 0100010011010101111

LOUDS – level order unary degree sequence

- Each node (excpet the pseudo root) is represented twice
  - Once as „0” in the child list of its parent
  - Once as the terminel („1”) in its child list
- Represent node *v* by the index of its corresponding „0”
- I.e. *root* corresponds to „0”

LOUDS – level order unary degree sequence

```
is_leaf(v)
    id ← rank(v, 0, LOUDS)
    p ← select(id + 1, 1, LOUDS)
    if p + 1 = LOUDS.size() or LOUDS[p + 1] = 1 then
        return true
    return false
```

```
out_degree(v)
    if is_leaf(v) then
        return 0
    id ← rank(v, 0, LOUDS)
    return select(id + 2, 1, LOUDS) − select(id + 1, 1, LOUDS) − 1
```

LOUDS – level order unary degree sequence

Get i-th child of v and parent:

child(v,i)
  if $i > out\_degree(v)$ **then**
    return $\perp$
  $id \leftarrow rank(v, 0, LOUDS)$
  return $select(id + 1, 1, LOUDS) + i$

parent(v)
if $is\_root(v)$ **then**
  return $\perp$
  $pid \leftarrow rank(v, 1, LOUDS)$
  return $select(pid, 0, LOUDS)$

LOUDS – level order unary degree sequence

Conclusion

- Total space for LOUDS representation is $2n + 1 + o(n)$ bits
- All operations take constant time