Advanced Data Structures
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Institute of Theoretical Informatics - Algorithmics
Range minimum queries (RMQs)

Definition
Given an array $A$ of length $n$ containing elements from a totally ordered set. A range minimum query $rmq_A(\ell, r)$ returns the position of the minimal element in the sub-array $A[\ell, r]$:

$$rmq_A(\ell, r) = \arg\min_{\ell \leq k \leq r} A[k]$$

Example

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 \end{bmatrix}$$
Range minimum queries (RMQs)

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\[
rmq_A(\ell, r) = \arg \min_{\ell \leq k \leq r} A[k]
\]

Example

\[
A = \begin{bmatrix}
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 \\
\end{bmatrix}
\]

\( rmq(2, 6) = 4 \)
Range minimum queries (RMQs)

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0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8
\end{bmatrix}$$

$$rmq(5, 9) = 6$$
Range minimum queries (RMQs)

Definition
Given an array $A$ of length $n$ containing elements from a totally ordered set. A range minimum query $rmq_A(\ell, r)$ returns the position of the minimal element in the sub-array $A[\ell, r]$: 

$$rmq_A(\ell, r) = \arg\min_{\ell \leq k \leq r} A[k]$$

Example

$$A = \begin{bmatrix}
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 \\
\end{bmatrix}$$

$\text{rmq}(0, 15) = 4$
Overview

- Notation: Complexity of an algorithm is denoted with \( \langle f(n), g(n) \rangle \), where \( f(n) \) is preprocessing time and \( g(n) \) query time.
- Different solutions:
  - naïve approach 1: \( \langle O(n^2), O(1) \rangle \) using \( O(n^2) \) words of space
  - naïve approach 2: \( \langle O(1), O(n) \rangle \)
  - \( \langle O(n), O(\log n) \rangle \) using \( O(n) \) words of space
  - \( \langle O(n \log n), O(1) \rangle \) using \( O(n \log n) \) words of space
  - \( \langle O(n \log \log n), O(1) \rangle \) using \( O(n \log \log n) \) words of space
  - \( \langle O(n), O(1) \rangle \) using \( O(n) \) words of space
  - \( \langle O(n), O(1) \rangle \) using \( 4n + o(n) \) bits of space
  - \( \langle O(n), O(1) \rangle \) using \( 2n + o(n) \) bits of space

Note
The last two solutions do not require access to the original array \( A \).
Overview

Literature

- M.A. Bender, M. Farach-Colton: The LCA Problem Revisited. (LATIN 2000)
- K. Sadakane: Compressed Suffix Trees with Full Functionality. (TCS 2007)
- H. Ferrada, G. Navarro: Improved Range Minimum Queries (DCC 2016)
\[ O(n), O(\log n) \] – solution #1

\[ A = \begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 
\end{array} \]
\( O(n), O(\log n) \) – solution #1

\[
\begin{array}{cccccccccccccc}
4 \\
4 & 13 \\
1 & 4 & 9 & 13 \\
1 & 2 & 4 & 6 & 9 & 11 & 13 & 14 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\end{array}
\]

\[
A = \begin{pmatrix}
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8
\end{pmatrix}
\]

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\( \langle O(n), O(\log n) \rangle \) – solution #1

\[ A = \]

\[
\begin{array}{cccccccccccccccc}
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 \\
\end{array}
\]

\[ rmq(1, 5) = 4 \]
\[ \langle O(n), O(\log n) \rangle \quad \text{– solution #1} \]

\[ A = \begin{bmatrix}
8 & 2 & 4 & 7 & 1 & 9 & 3 & 5 & 7 & 4 & 6 & 4 & 3 & 1 & 4 & 8 \\
\end{bmatrix} \]

\[ \text{rmq}(1, 5) = 4 \]
Store index of minimum in binary interval tree.
- Tree has $O(n)$ nodes.
- Follow all nodes which overlap with the query interval but are not fully contained in it (at most 2 per level).
- So not more than $2 \log n$ such nodes in total.
- Select all children of these nodes which are fully contained in the query interval.
- From these nodes select the index with minimal value.
\( O(n\log n), O(1) \) – solution #2

- For each item \( A[i] \) store an array \( M_i[0, \log n] \).
- \( M_i[j] = \text{rmq}_A(i, i + 2^j - 1) \)
- Space is \( O(n\log n) \) words
- How long does pre-computation take?

**Querying**

Find the largest \( k \) with \( 2^k \leq \ell - r + 1 \). Then

\[
\text{rmq}_A(\ell, r) = \begin{cases} 
M_i[k] & \text{if } A[M_i[k]] < A[M_{j-2^k+1}[k]] \\
M_{j-2^k+1}[k] & \text{otherwise}
\end{cases}
\]

Question: How can \( k \) be determined in constant time?
\[ \langle O(n \log \log n), O(1) \rangle \text{ solution} \]

- Split \( A \) into \( t = \frac{n}{\log n} \) blocks \( B_0, \ldots, B_{t-1} \). \( B \) spans \( O(\log n) \) items of \( A \).
- Create an array \( S[0, t - 1] \) with \( S[i] = \min\{x \in B_i\} \)
- Build rmq structure #2 for \( S \)
- For each block \( B_i \) of \( O(\log n) \) elements build rmq structure #2
- Total space: \( O(n) + O(n \log \log n) \)

**Querying**

- Determine blocks \( B_{\ell'}, B_{r'} \) which contain \( \ell \) and \( r \)
- Calculate \( m = \text{rmq}_S(\ell' + 1, r' - 1) \)
- Let \( k_0, k_1, k_2 \) be the results of the RMQs in blocks \( \ell', r', \) and \( m \) relative to \( A \)
- Return \( \arg \min_{k_i} A[k_i] \) for \( 0 \leq k \leq 3 \)
Definition

The Cartesian Tree \( C \) of an array is defined as follows:

- The root of \( C \) is the (leftmost) minimum element of the array and is labeled with its position.
- Removing the root splits the array into two pieces.
- The left and right children of the root are recursively constructed Cartesian trees of the left and right subarray.
- \( C \) can be constructed in linear time.
\(O(n), O(1)\) solution

Solution overview:

- Partition the array into blocks of size \(s\)
- Each block corresponds to a Cartesian Tree of size \(s\)
- Precompute the \(s^2\) answers for all \(\frac{1}{n+1}\binom{2n}{n}\) possible Cartesian Trees of size \(s\) in a table \(P\).
- \(P\) requires \(O(2^{2s}s^2)\) words of space
- For \(s = \frac{\log n}{4}\) \(P\) requires \(o(n)\) words of space
- Build structure #2 for array \(A'\) consisting of the block minima of \(A\). This takes \(O(n)\) construction time and uses \(O(n)\) words of space.
LCA (Lowest Common Ancestor)

Given a rooted tree $T$ of $n$ nodes. For nodes $v$ and $w$ of $T$ the query $LCA_T(v, w)$ returns the *lowest common ancestor* of $u$ and $v$ in $T$. I.e. the node which is (1) ancestor of $v$ and $w$ and (2) maximizes the distance to the root.
LCA (Lowest Common Ancestor)

Given a rooted tree $T$ of $n$ nodes. For nodes $v$ and $w$ of $T$ the query $LCA_T(v, w)$ returns the \textit{lowest common ancestor} of $u$ and $v$ in $T$. I.e. the node which is (1) ancestor of $v$ and $w$ and (2) maximizes the distance to the root.
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LCA & ±1 RMQ

Lemma
If there is a \( \langle f(n), g(n) \rangle \)-time solution for RMQ, then there is a \( \langle f(2n - 1) + O(n), g(2n - 1) + O(1) \rangle \)-time solution for LCA.

- Let \( T \) be the Cartesian Tree of array \( A \)
- Let array \( E[0, \ldots, 2n - 2] \) store the nodes visited in an DFS Euler Tour of \( T \)
- Let array \( L[0, \ldots, 2n - 2] \) store the corresp. levels of the nodes in \( E \)
- Let \( R[0, \ldots, n - 1] \) be an array which stores a representative \( R[i] = \min\{j \mid E[j] = i\} \) for each node \( i \) of \( T \)
Example

```
i = 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0
E = a b c b d b a e f g f e h e i j k i e a
L = 0 1 2 1 2 1 0 1 2 3 2 1 2 1 2 3 2 3 2 1 0
R = 0 1 2 4 7 8 9 12 14 15 17
```
LCA & ±1RMQ

Example

\[
i = 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0
\]
\[
E = a b c b d b a e f g f e h e i j i k i e a
\]
\[
L = 0 1 2 1 2 1 0 1 2 3 2 1 2 1 2 3 2 3 2 1 0
\]
\[
R = 0 1 2 4 7 8 9 12 14 15 17
\]

\[
LCA_T(v, w) = E[RMQ_L(min(R[v], R[w]), max(R[v], R[w]))]
\]
Example

\[ i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0\} \]
\[ E = \{a, b, c, b, d, b, a, e, f, g, f, e, h, e, i, j, i, k, i, e, a\} \]
\[ L = \{0, 1, 2, 1, 2, 1, 0, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 3, 2, 1, 0\} \]
\[ a, b, c, d, e, f, g, h, i, j, k \]
\[ R = \{0, 1, 2, 4, 7, 8, 9, 12, 14, 15, 17\} \]

\[ LCA_T(v, w) = E[\text{RMQ}_L(\min(R[v], R[w]), \max(R[v], R[w])))] \]

Note: \((L[i] - L[i + 1]) \in \{-1, +1\}\). So we only need to solve RMQs over arrays with this ±1 restriction. This is called ±1 RMQ.
\[ \langle O(n), O(1) \rangle \text{ solution } (4n + o(n) \text{ bits}) \]

\[
\begin{array}{cccccccccccc}
  &  &  &  &  &  &  &  &  &  &  & 1 \\
 1 & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 \\
 A & = & 4 & 6 & 3 & 5 & 1 & 4 & 6 & 4 & 5 & 2 & 6 & 3
\end{array}
\]
$\langle O(n), O(1) \rangle$ solution ($4n + o(n)$ bits)

$i = 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 0\ 1$

$A = 4\ 6\ 3\ 5\ 1\ 4\ 6\ 4\ 5\ 2\ 6\ 3$

(1) Build Cartesian Tree
\[ \langle O(n), O(1) \rangle \text{ solution } (4n + o(n) \text{ bits}) \]

\[ i = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 0 \ 1 \]

\[ A = 4 \ 6 \ 3 \ 5 \ 1 \ 4 \ 6 \ 4 \ 5 \ 2 \ 6 \ 3 \]

(1) Build Cartesian Tree
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\[
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\end{align*}
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\end{align*}
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\end{align*}
\]

(2) Add a leaf to each node
\[ \langle O(n), O(1) \rangle \text{ solution } (4n + o(n) \text{ bits}) \]

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{\text{ext}} = \]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = (1
\]
\[\langle O(n), O(1) \rangle \] solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

$$
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
$$

(3) DFS traversal to construct balanced parentheses sequence

\[BP_{\text{ext}} = ()()\]
\langle O(n), O(1) \rangle \textbf{ solution } (4n + o(n) \text{ bits})

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{\text{ext}} = ( ( ( ) ) ) \]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[ BP_{ext} = ( ( ( (} \]
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BP_{\text{ext}} = ( ((()())( 0
\]

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\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[
BP_{\text{ext}} = (((()())())
\]

0 1 2 3 4 5 6 7 8 9 10 11
\(O(n), O(1)\) solution \((4n + o(n)\) bits\)

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\[BP_{ext} = ((())(()))\]
\( \langle O(n), O(1) \rangle \) solution (\( 4n + o(n) \) bits)

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\[
BP_{ext} = (((()())()))(0 1)
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

\( \begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array} \)

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{\text{ext}} = ( ((()(())))() ) \]

0 1 2
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = ( (( () ( )) ) ( ) ( 0 1 2 3 4 5 6 7 8 9 10 11 )
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{ext} = ( ((())())())()() \]

0 1 2 3 4 5 6 7 8 9 10 11
\( \langle O(n), O(1) \rangle \) **solution** \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{ext} = ( ((()(())))(())() ) \]

0 1 2 3 4 5 6 7 8 9 10 11
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(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{\text{ext}} = ( ((()(()))())(()(())) ) \]

\[ \begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array} \]
\(\langle O(n), O(1) \rangle\) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = \left( ( ( ( ) ( ( ) ) ) ( ) ( ( ) ) ) \right) ( \\
0 & 1 & 2 & 3
\right)
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[
BP_{\text{ext}} = ( ( ( ( ) ( ( ) ) ) ( ) ( ( ) ) ) ( )
\]

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\[
BP_{ext} = ( ( ( ) ( ( ) ) ) ( ) ( ( ) ) ) ( ) \\
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[
BP_{\text{ext}} = (((()())())())()()()()
\]
\[ O(n), O(1) \] solution \((4n + o(n) \text{ bits})\)

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BP_{ext} = (()(()())())()()()()()()
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0 1 2 3 4 5 6 7 8 9 10 11

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\[ \langle O(n), O(1) \rangle \textbf{ solution} \ (4n + o(n) \text{ bits}) \]

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[ BP_{\text{ext}} = \langle ((())((()))((()))(())(()(())) \rangle \]

0 1 2 3 4 5 6 7 8 9 10 11
\( \langle O(n), O(1) \rangle \) solution \( (4n + o(n) \text{ bits}) \)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{ext} = ( (((())(()))())(())(((())())())()
\]

1 3 4 6 6 4 5 5 3 2 4 4
\(\langle O(n), O(1)\rangle\) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

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\]

0 1 2 3 4 5 6 7 8 9 10 11
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\[
BP_{\text{ext}} = ( ((()(()))())(()())())(()())(()())
\]
\[\langle O(n), O(1) \rangle \text{ solution } (4n + o(n) \text{ bits})\]

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BP_{\text{ext}} = ((())((()))())(()())(()())(()())
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{ext} = (((()((())))())())())(()(()(()(())))(
\]

0 1 2 3 4 5 6 7 8 9 10 11
\langle O(n), O(1) \rangle \textbf{ solution } (4n + o(n) \text{ bits})

(2) Add a leaf to each node

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$$BP_{ext} = ( ((())())())(()(()))())(()(()))(()))()$$

0 1 2 3 4 5 6 7
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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BP_{ext} = ( ((()) ((())) (()) )) (()) ((()) ((())) (()) )
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(2) Add a leaf to each node

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0 1 2 3 4 5 6 7 8 9 10 11
```

(3) DFS traversal to construct balanced parentheses sequence

\[ \text{BP}_{\text{ext}} = (((()(())())((())())(()(())())(()()))) \]

```
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[ \begin{align*}
BP_{ext} &= (( ( ( ( ) ) ) ) ( ) ( ( ) ) ) ( ) ( ( ) ) ( ( ) ) ( ) ( ( ) )
\end{align*} \]
\[ \langle O(n), O(1) \rangle \] solution \((4n + o(n)\) bits) 

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\[ BP_{ext} = ((())()())(())(())(())((())())()()() \]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

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\[ BP_{ext} = ( ((()(()))) (()(())) () ((((()(()))))) ) ) \]

0 1 2 3 4 5 6 7 8 9 10 11
\[ \langle O(n), O(1) \rangle \text{ solution (} 4n + o(n) \text{ bits)} \]

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\[ BP_{\text{ext}} = (\((\((\(())(())\))\))\((\()\((())\))\))\((\((())((())(()))\))\))\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = ( (( ( ) ( ( ) ) ) ( ) ( ( ) ) ) ( ) ( ( ) ( ( ) ) ( ) ( ( ) ) ) ) ( 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8
\]
\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = ( ((()()())())()()())(()())(()())(()())(()())(()())(()())() \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\]
\langle O(n), O(1) \rangle \textbf{ solution } (4n + o(n) \text{ bits})

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0 1 2 3 4 5 6 7 8 9 10 11
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\[ BP_{ext} = \left( ( ( ( ) ( ( ) ) ) ( ) ( ( ) ) ) ( ) ( ( ) ( ( ) ) ( ) ( ( ) ) ) ) ( ) ( \right) \]

\[ \downarrow \]

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0 1 2 3 4 5 6 7 8 9
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Simon Gog: Advanced Data Structures

Institute of Theoretical Informatics
Algorithmics
\(O(n), O(1)\) solution \((4n + o(n) \text{ bits})\)

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\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

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\( \langle O(n), O(1) \rangle \) solution \((4n + o(n) \text{ bits})\)

(2) Add a leaf to each node

(3) DFS traversal to construct balanced parentheses sequence

\[
BP_{\text{ext}} = \begin{array}{c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & 3 & 4 & 6 & 6 & 4 & 5 & 5 & 3 & 2 & 4 & 4 & 6 & 6 & 5 & 5 & 4 & 4 & 3 & 6 \\
\end{array}
\]
(2) Add a leaf to each node

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\[ BP_{\text{ext}} = ( ( ( ( ( ( ( ( ) ) ) ) ) ) ( ( ( ( ( ( ( ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) ) }
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\[ BP_{ext} = ( ((())(())) (())(())) () ((())(())) () ((())(())) () ((())(())) () ((())(())) ( \]

1 3 4 6 6 4 5 5 3 2 4 4 6 6 5 5 4 4 3 6 6
\[ \langle O(n), O(1) \rangle \text{ solution } (4n + o(n) \text{ bits}) \]

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\[ BP_{\text{ext}} = ( ((()((())))())(())())(((())((())())())())(()(()))() \]
\(\langle O(n), O(1)\rangle\) solution \((4n + o(n)\) bits\)

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\[BP_{ext} = ( ( ( ( ) ( ( ) ) ) ) ( ) ( ( ) ) ) ( ) ( ( ( ) ( ( ) ) ( ) ( ( ) ) ) ( ) ( ( ) ) ( ) ) \]

0 1 2 3 4 5 6 7 8 9 10 11
\( \langle O(n), O(1) \rangle \) \textbf{solution} \((4n + o(n) \text{ bits})\)

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\begin{array}{cccccccccccc}
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\end{array}
\]

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BP_{ext} = ( (((())(())())(())())()) (((())((())())())()) ) ( (((())())())())
\]
\( \langle O(n), O(1) \rangle \) solution \( (4n + o(n) \text{ bits}) \)

- The extended Cartesian Tree contains \( 2n \) nodes
- The balanced parentheses sequence consists of \( 4n \) bits
- The position of each leaf corresponds to the inorder number of its parent in the Cartesian tree
- The inorder number corresponds to the array index of the element
- Let \( \text{excess}(i) = \text{rank}(i + 1, 1, \text{BP}_{\text{ext}}) - \text{rank}(i + 1, 0, \text{BP}_{\text{ext}}) \)

For \( 0 \leq i \leq j < n \) we get:

00 \( \text{rmq}_A(i, j) \)

01 \( \text{ipos} \leftarrow \text{select}(i + 1, (\), \text{BP}_{\text{ext}}) \)

02 \( \text{jpos} \leftarrow \text{select}(j + 1, (\), \text{BP}_{\text{ext}}) \)

03 \( \text{return} \ \text{rank}(\text{rmq}_{\text{excess}}^{\pm 1}(\text{i pos}, \text{j pos} + 1), (\), \text{BP}_{\text{ext}}) \)
Added leaves are used to navigate to inorder index nodes
Select on a pattern „()” of fixed size can be done in constant time after precomputing a o(n)-space structure
Let \( v \) be the \((i + 1)\)th leaf node
Let \( w \) be the \((j + 1)\)th leaf node
\( \text{rmq}_{\text{excess}}^{\pm 1}(ipos, jpos + 1) \) is the position of closing parenthesis of the leaf node \( z \) added to \( \text{LCA}(v, w) \)
In-order number of \( \text{LCA}(v, w) \) corresponds to index of minimum in \( A[i, j] \) and can be determined by a rank operation on pattern „( )”

Next: \( o(n) \) data structure to support \( \text{rmq}_{\text{excess}}^{\pm 1} \) queries on \( BP_{\text{ext}} \)
\[ O(n), O(1) \] solution \((4n + o(n) \text{ bits})\)

- Divide the (conceptional) array excess in blocks of size \(\log^3 n\)
- \(S[0, n/\log^3 n]\) stores the minima of the blocks
- Solution #2 for \(S\) requires \(O\left(\frac{n}{\log^3 n} \cdot \log^2 n\right) = o(n)\) bits
- Divide each block in subblocks of size \(\frac{1}{2} \log n\)
- Apply again solution #2 on subblocks \((n' = \log^2 n)\). I.e. total space \(O\left(\frac{n}{\log n} \log n' \cdot \log n'\right) = O\left(\frac{n}{\log n} \log^2 \log n\right) = o(n)\)
- Lookup table for blocks of size \(\frac{1}{2} \log n\) is also in \(o(n)\)
\( \langle O(n), O(1) \rangle \) solution \((2n + o(n) \text{ bits})\)

Observation
Adding the leaf nodes enables inorder indexing of the nodes but doubles the space.

Next: Approach which does not require additional nodes.
- Cartesian Tree (CT) is a binary tree of \( n \) nodes
- Transform CT into general tree of \( n + 1 \) nodes
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\( \langle O(n), O(1) \rangle \) \textbf{solution} \((2n + o(n) \text{ bits})\)

Transformation

- Add a new root node to the leaf of the original root
- For each node \( v \) (starting at the root) take the right child \( w \), and add the nodes on the leftmost path from \( w \) as children of \( v \)
$\langle O(n), O(1) \rangle$ solution $(2n + o(n) \text{ bits})$

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\( \langle O(n), O(1) \rangle \)  \textbf{solution} \((2n + o(n) \text{ bits})\)

Transformation

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- Node with inorder \(i\) in CT becomes node with preorder \(i + 1\) in the general tree
- So we can identify the node of array element \(i\) by selecting the \((i + 2)\)th opening parenthesis in the balanced parentheses sequence of the general tree (+1 for the added root node, +1 for index shift by one)
- Let \(BP\) be the balanced parentheses sequence of the general tree

For \(0 \leq i < j < n\) we get:

00 \(\text{rmq}_A(i, j)\)
01 \(\text{iros} \leftarrow \text{select}(i + 2, (, BP)\)
02 \(\text{jpos} \leftarrow \text{select}(j + 2, (, BP)\)
03 \(\text{return rank}(\text{rmq}_{\text{excess}}^{\pm 1}(\text{iros} - 1, \text{jpos}), (, BP) - 1\)

Where \(\text{rmq}_{\text{excess}}^{\pm 1}\) returns the rightmost position of the minimal value.
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Proof sketch: Let \( v \) and \( w \) be the nodes of \( A[i] \) and \( A[j] \) and \( z \) be the LCA of \( v \) and \( w \) in the Cartesian Tree. Node \( v \) is in the left subtree and \( w \) in the right subtree of \( z \).

Situation in binary tree

Situation in general tree
\( \langle O(n), O(1) \rangle \) solution \((2n + o(n) \text{ bits})\)

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Situation in binary tree

Situation in general tree
\[ \langle O(n), O(1) \rangle \text{ solution } (2n + o(n) \text{ bits}) \]

First case: \( v \neq z \) and \( w \neq z \)

- \( v \)'s opening parenthesis is the green area
- \( w \)'s opening parenthesis is the red area
- the (rightmost) \( \pm 1 \) RMQ will return the position \( p \) of the closing parenthesis of \( z \)
- \( z \)'s opening parenthesis is at position \( p + 1 \) by construction
- \( r = rank(p) - 1 \) corresponds to the preorder number of \( z \) in the general tree
- which in turns corresponds to the inorder number in \( CT \)
- which in turns corresponds to the index of the minimum in \( A[i, j] \)
Second case: $v \neq z$ and $w = z$:

- $v$’s opening parenthesis is the green area
- $w$’s opening parenthesis is $z$’s opening parenthesis now
- The (rightmost) $\pm 1$ RMQ will return the position $p$ of the closing parenthesis of $z$.
- $z$’s opening parenthesis is at position $p + 1$ by construction
- $r = rank(p) - 1$ corresponds to the preorder number of $z$ in the general tree
- Which in turns corresponds to the inorder number in CT
- Which in turns corresponds to the index of the minimum in $A[i, j]$
Third case: $v = z$ and $w \neq z$:
- $v$’s opening parenthesis is $z$’s opening parenthesis now
- $w$’s opening parenthesis is the red area
- The (rightmost) ±1 RMQ will return the position $p$ of the closing parenthesis of $z$
- $z$’s opening parenthesis is at position $p + 1$ by construction
- $r = \text{rank}(p) - 1$ corresponds to the preorder number of $z$ in the general tree
- which in turns corresponds to the inorder number in CT
- which in turns corresponds to the index of the minimum in $A[i, j]$
$\langle O(n), O(1) \rangle$ solution ($2n + o(n)$ bits)

Third case: $v = z$ and $w = z$:

- $v$’s opening parenthesis is $z$’s opening parenthesis now
- $w$’s opening parenthesis is $z$’s opening parenthesis now
- the (rightmost) $\pm 1$ RMQ will return the position $p$ of the closing parenthesis of $z_i$
- $z$’s opening parenthesis is at position $p + 1$ by construction
- $r = \text{rank}(p) - 1$ corresponds to the preorder number of $z$ in the general tree
- which in turns corresponds to the inorder number in CT
- which in turns corresponds to the index of the minimum in $A[i, j]$
Range Minimum Queries (RMQ)s over an array $A$ can be answered in constant time after preprocessing a $2n + o(n)$ space data structure in linear time.

Applications:
- Compressed suffix trees
- Document retrieval
- Weighted query completion
- …