Voronoi Diagrams

- *site* = element of our ground set S
- for any $p \in \mathbb{R}^2$: $close(p)$ = the set of sites that realize the closest distance between p and the sites in S.
- $VD(S) = \{p \in \mathbb{R}^2; |close(p)| \geq 2\}$
- $VD(S)$ is a graph like structure
  - vertices = points p with $|close(p)| \geq 3$,
  - edges = maximal connected sets of points p with $|close(p)| = 2$,
  - faces = maximal connected sets of points p with $|close(p)| = 1$.

Characterization of Edges

Lemma 2 *Edges are parts of perpendicular bisectors of points in S*

- consider any edge e of the Voronoi diagram, and let s and t be the two sites of S such that $close(p) = \{s, t\}$ for all points p of e.
- any such p lies on the perpendicular bisector of s and t and hence e is a straight line segment contained in the perpendicular bisector of s and t.

Characterization of Regions (Faces)

- $VR(s,t)$ = open half-plane bounded by the perpendicular bisector of s and t and containing s.
- consider any face f of the Voronoi diagram and let s be the site of S such that $close(p) = \{s\}$ for all points p of f.
- then $dist(s,p) < dist(t,p)$ for all $t \in S \setminus \{s\}$ and hence $f \subseteq V(s,t)$.
- thus $f \subseteq VR(s) := \bigcap_{t \in S \setminus \{s\}} VR(s,t)$
- we have equality since $p \in VR(s)$ implies $p \in VR(s,t)$ for all $t \in S \setminus \{s\}$ which in turn implies that p is closer to s than to any other site in S.
- $VR(s)$ = *Voronoi region* of site s.
- $VR(s)$ is the intersection of open halfspaces and hence an open convex polygonal region.

Duality between Delaunay Diagrams and Voronoi Diagrams

Theorem 2 *The following rules construct the Voronoi diagram $VD(S)$ of a point set S from its Delaunay diagram $DD(S)$.*

(1) For every bounded face f of $DD(S)$ there is a vertex $c(f)$ of $VD(S)$ located at the center of the circumcircle of f.

(2) Consider an edge st of $DD(S)$ and let f_1 and f_2 be the faces incident to the two sides of the edge.
(a) If f_1 and f_2 are both bounded, then the edge $c(f_1)c(f_2)$ belongs to $VD(S)$.
(b) If f_1 is unbounded and f_2 is bounded, then a ray with source $c(f_2)$ and contained in the perpendicular bisector of s and t belongs to $VD(S)$. It extends into the half-plane containing the unbounded face.
(c) If f_1 and f_2 are unbounded[a] and hence $f_1 = f_2$, then the entire perpendicular bisector of s and t belongs to $VD(S)$.

(3) That's all.

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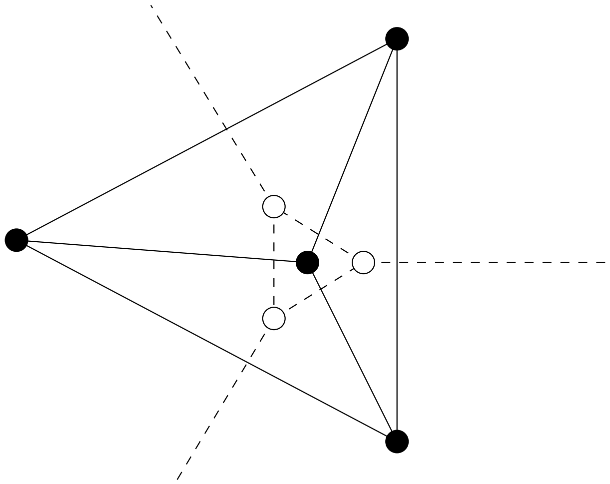
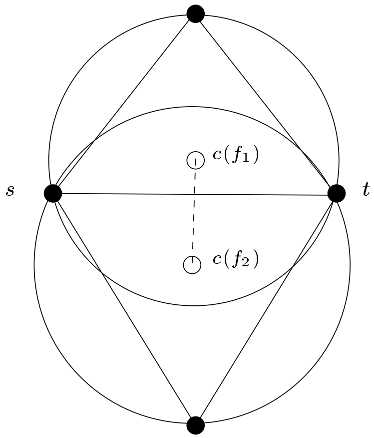
[a] Case (c) arises only if all sites in S are collinear. Then cases (a) and (b) do not arise.

Illustration of Duality

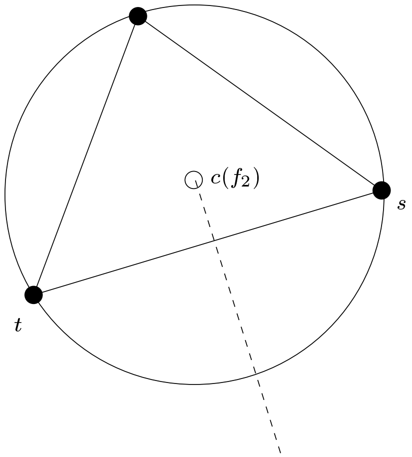


Correctness I, only parts of $VD(S)$ are constructed

- consider any bounded face f of $DD(S)$. The vertices of f are co-circular and hence $c(f)$ is a point with $|close(p)| \geq 3$, i.e., a vertex of $VD(S)$.
- consider next any edge st of $DD(S)$. View it as oriented from s to t and let f_1 and f_2 be the faces to its left and right, respectively. Assume first that f_1 and f_2 are both bounded. The centers $c(f_1)$ and $c(f_2)$ of the circumcircles of f_1 and f_2 both lie on the perpendicular bisector of s and t and any point between $c(f_1)$ and $c(f_2)$ is the center of a disk D with $D \cap S = \{s,t\}$. Thus, $c(f_1)c(f_2)$ is an edge of $VD(S)$.



Assume next that f_1 is unbounded and f_2 is bounded, i.e, st is a clockwise convex hull edge.
The ray starting in $c(f_2)$, contained in the perpendicular bisector of s and t, and extending into the left half-plane with respect to st belongs to $VD(S)$.



Finally, if f_1 and f_2 are both unbounded then the entire perpendicular bisector of s and t is an edge of $VD(S)$.

All of $VD(S)$ is constructed

- consider any edge e of $VD(S)$, say separating the regions $VR(s)$ and $VR(t)$.
- then $close(p) = \{s,t\}$ for every point $p \in e$, i.e., every $p \in e$ witnesses that the segment st is essential and hence is an edge of $DD(S)$.
- imagine a disk centered at p and having s and t in its boundary as p moves along e. When p moves into an endpoint of e (e may have 0, 1, or 2 endpoints), $close(p)$ grows to at least three points, namely the vertices of a face of $DD(S)$ incident to st.
- thus, applying the appropriate rule 2a, 2b, or 2c to st yields e.
- moreover, applying rule 1 to the bounded faces incident to st produces the endpoints of e (if any).
- we have now shown that all edges of $VD(S)$ are constructed and since every vertex of $VD(S)$ is incident to at least one (actually three) edge we have also shown that all vertices are constructed.