Sorting Strings and Suffixes

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Overview

- String sorting (from a mini course at MPII by Juha)
- Skew: Simple scalable suffix sorting
  (also at ICALP 2003 (July) Eindhoven, Netherlands)
- More

String sorting problem

Sort a set $R = \{s_1, s_2, \ldots, s_n\}$ of n (non-empty) strings into the lexicographic order.

Size of input

- $N =$ total length of strings
- $D =$ total length of distinguishing prefixes

Some Notation:

- $s = s[0] \ldots s[|s|-1]$
- $\forall c \in \Sigma : s[|s|] > c$ (special sentinel character)

Distinguishing prefix

The distinguishing prefix of string s in R is

- shortest prefix of s that is not a prefix of another string (or s if s is a prefix of another string)
- shortest prefix of s that determines the rank of s in R

```
alignment
all
allocate
alphabet
alternate
alternative
```

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A sorting algorithm needs to access

- every character in the distinguishing prefixes
- no character outside the distinguishing prefixes

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Alphabet model

Ordered alphabet

- only comparisons of characters allowed

Constant alphabet

- ordered alphabet of constant size
- multiset of characters can be sorted in linear time

Integer alphabet

- alphabet is $\{1,\ldots,\sigma\}$ for integer $\sigma \geq 2$
- multiset of k characters can be sorted in $O(k+\sigma)$ time

Lower bounds

alphabet	lower bound
ordered	$\Omega(D + n \log n)$
constant	$\Omega(D)$
integer	$\Omega(D)$

Standard sorting algorithm

- $\Theta(n \log n)$ string comparisons

Let $s_i = \alpha\beta_i$, where $|\alpha| = |\beta_i| = \log n$

- $D = \Theta(n \log n)$
- lower bound:
  $\Omega(D + n \log n) = \Omega(n \log n)$
- standard sorting:
  $\Theta(n \log n) \cdot \Theta(\log n) = \Theta(n \log^2 n)$

aaaaaa	aaaaab
aaaaba	aaaabb
aaabaa	aaabab
aaabba	aaabbb

Multikey quicksort *[Bentley & Sedgewick ’97]*

- ternary partition
- on one character at a time

$<$	$=$	$>$

al	p	habet		al	i	gnment
al	i	gnment		al	g	orithm
al	l	ocate		al	i	as
al	g	orithm	$\Longrightarrow$	al	l	ocate
al	t	ernative		al	l	
al	i	as		al	p	habet
al	t	ernate		al	t	ernative
al	l			al	t	ernate

Multikey quicksort *[Bentley & Sedgewick '97]*

Multikey-quicksort(R, ℓ) // $R =$ set of strings with
// common prefix of length ℓ

1 if $|R| \leq 1$ then return R

2 choose pivot $p \in R$

3 $R_< := \{s \in R \mid s[\ell+1] < p[\ell+1]\}$
$R_= := \{s \in R \mid s[\ell+1] = p[\ell+1]\}$
$R_> := \{s \in R \mid s[\ell+1] > p[\ell+1]\}$

4 Multikey-quicksort($R_<$, ℓ)
5 Multikey-quicksort($R_=$, $\ell+1$)
6 Multikey-quicksort($R_>$, ℓ)

7 return $R_< R_= R_>$

Multikey quicksort: Analysis

- comparisons in partitioning step dominate runtime

```
if |R| ≤ 1 then return R
choose pivot p ∈ R
R< := {s ∈ R | s[ℓ+1] < p[ℓ+1]}
R= := {s ∈ R | s[ℓ+1] = p[ℓ+1]}
R> := {s ∈ R | s[ℓ+1] > p[ℓ+1]}
Multikey-quicksort(R<, ℓ)
Multikey-quicksort(R=, ℓ+1)
Multikey-quicksort(R>, ℓ)
return R< R= R>
```

Multikey quicksort: Analysis

- If $s[\ell+1] \neq p[\ell+1]$, charge the comparison on s
  - assume perfect choice of pivot
  - size of the set containing s is halved
  - total charge on s is $\leq \log n$
  - total number of $\neq$-comparisons is $\leq n \log n$

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Multikey quicksort: Analysis

- If $s[\ell+1] = p[\ell+1]$, charge the comparison on $s[\ell+1]$
  - $s[\ell+1]$ becomes part of common prefix
  - total charge on $s[\ell+1]$ is ≤ 1
  - total number of $=$-comparisons is $\leq D$

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  - total charge on $s[\ell+1]$ is ≤ 1
  - total number of $=$-comparisons is $\leq D$
- $O(D + n \log n)$ time

Multikey quicksort: Analysis for Random Pivot

The analysis from standard sorting can be adapted to show that the expected number of $\neq$ comparisons is

$$2n \ln n$$

Multikey quicksort

alphabet	lower bound	upper bound	algorithm
ordered	$\Omega(D + n\log n)$	$O(D + n\log n)$	multikey quicksort
constant	$\Omega(D)$		
integer	$\Omega(D)$		

Radix sort

LSD-first radix sort

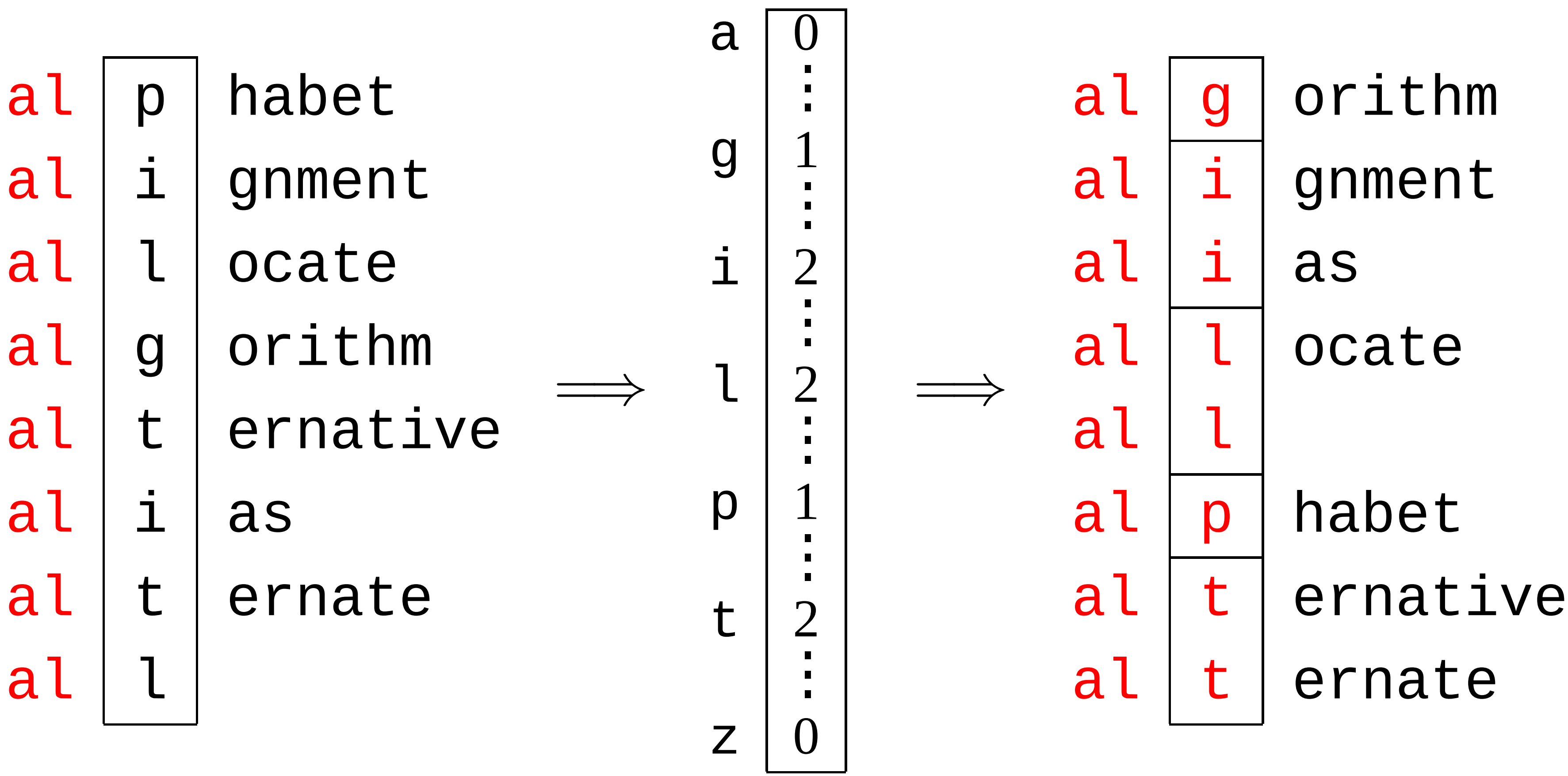
- starts from the end of the strings (Least Significant Digit first)
- accesses all characters: $\Omega(N)$ time
- poor when $D \ll N$

MSD-first radix sort

- starts from the beginning of the strings (Most Significant Digit first)
- accesses only distinguishing prefixes

(MSD-first) Radix sort

- recursive σ-way partitioning using counting sort



Radix sort: Analysis

- partitioning a group of k string takes $O(k+\sigma)$ time
- total size of the partitioned groups is D
- $O(D)$ total time on constant alphabets

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- do trivial partitioning (all characters are the same) in $O(k)$ time
- total number of non-trivial partitionings is $< n$
- $O(D+n\sigma)$ total time on integer alphabets
- switch to multikey quicksort when $k < \sigma$:
  $O(D+n\log\sigma)$ total time

Radix sort

alphabet	lower bound	upper bound	algorithm
ordered	$\Omega(D + n\log n)$	$O(D + n\log n)$	multikey quicksort
constant	$\Omega(D)$	$O(D)$	radix sort
integer	$\Omega(D)$	$O(D + n\log\sigma)$	radix sort + multikey quicksort

More radix sorts

alphabet	lower bound	upper bound	algorithm
ordered	$\Omega(D+n\log n)$	$O(D+n\log n)$	mk quicksort
constant	$\Omega(D)$	$O(D)$	radix sort
integer	$\Omega(D)$	$O(D+n\log\sigma)$	radix sort + mk quicksort
		$O(D+\sigma\log\sigma)$	breadth-first radix sort + mk quicksort
		$O(D+\sigma)$	two-pass radix sort

Suffix sorting problem

Sort the set $\{S_0, S_1, \ldots, S_{n-1}\}$ of the suffixes of a string S of length n (alphabet $[1,n] = \{1, \ldots, n\}$) into the lexicographic order.

- suffix $S_i = S[i,n]$ for $i \in [0 : n-1]$

$$S = \text{banana}$$

0	banana		5	a
1	anana		3	ana
2	nana	$\Longrightarrow$	1	anana
3	ana		0	banana
4	na		4	na
5	a		2	nana

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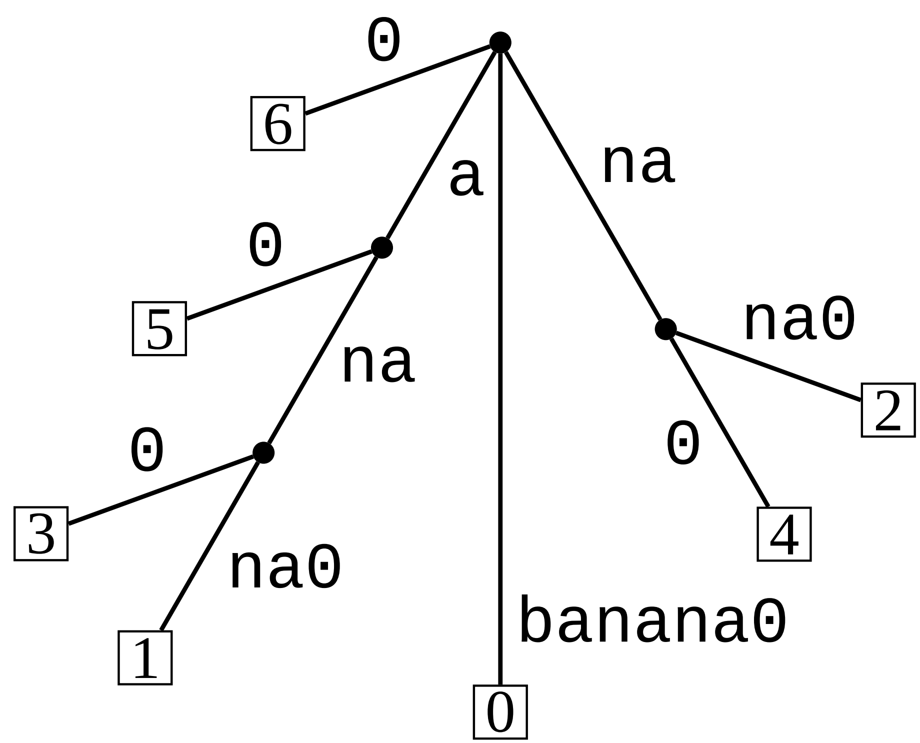
Applications

- full text indexing (binary search)
- Burrows-Wheeler transform (`bzip2` compressor)
- replacement for more complex suffix tree

Suffix tree *[Weiner '73][McCreight '76]*

- compact trie of the suffixes
- \+ $O(n)$ time [Farach 97] for integer alphabets
- \+ Most potent tool of stringology?
- − Space consuming
- − Efficient construction is compli-cated

$S =$ banana0



A First Divide-and-Conquer Approach

1. $SA^1 =$sort $\{S_i : i$ is odd$\}$ (recursion)
2. $SA^0 =$sort $\{S_i : i$ is even$\}$ (easy using SA^1)
3. merge SA^1 and SA^2 (very difficult)

Problem: its hard to compare odd and even suffixes.
[Farach 97] developed a linear time suffix tree construction algorithm based on that idea. Very complicated.

Was only known linear time algorithm for suffix arrays

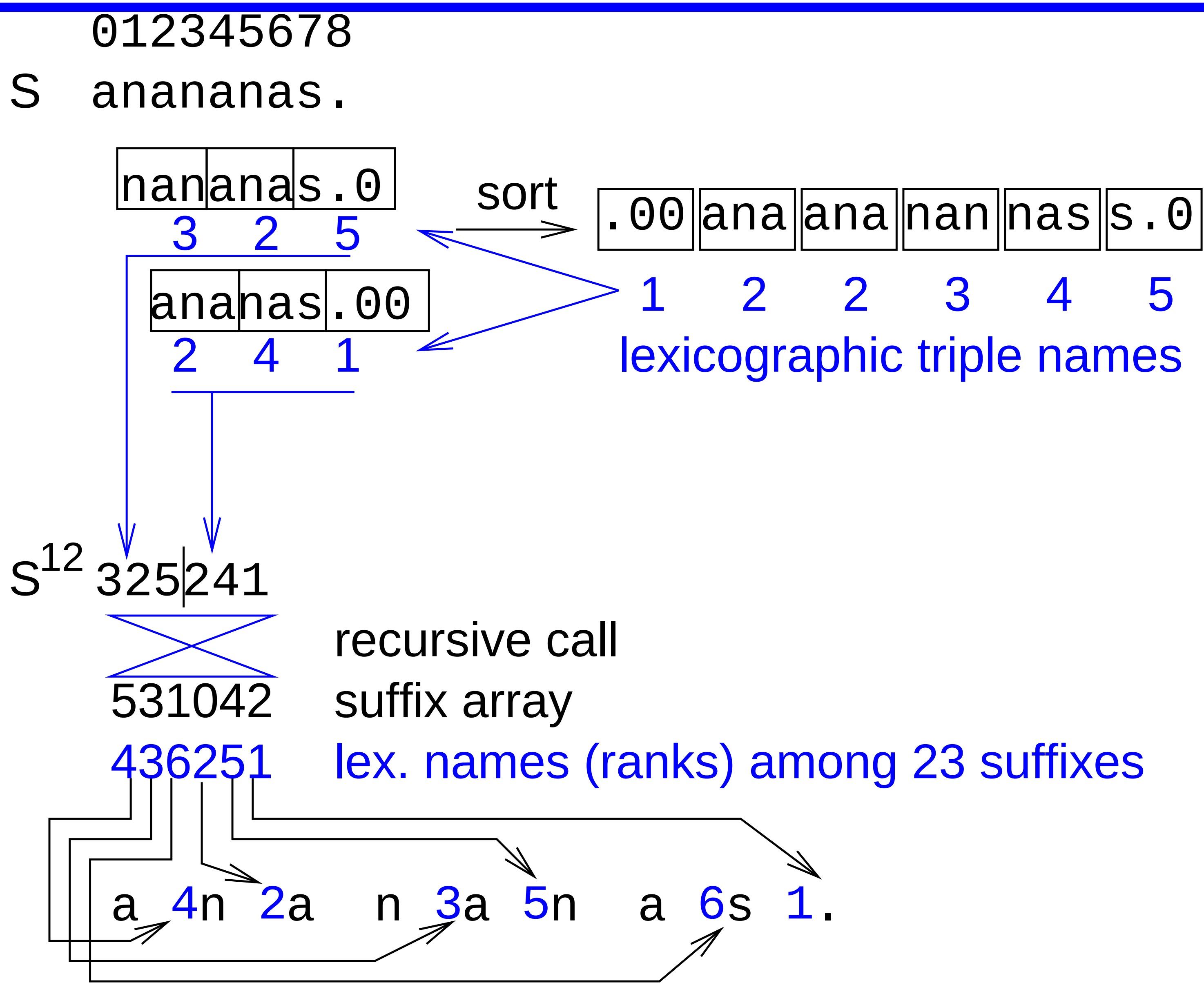
Skewed Divide-and-Conquer

1. $SA^{12} =$sort $\{S_i : i \bmod 3 \neq 0\}$ (recursion)
2. $SA^{0} =$sort $\{S_i : i \bmod 3 = 0\}$ (easy using SA^{12})
3. merge SA^{12} and SA^{0} (easy!)

$$S = \text{banana}$$

$$\begin{array}{|c|l} 5 & \text{a} \\ & \\ 1 & \text{anana} \\ & \\ 4 & \text{na} \\ 2 & \text{nana} \end{array} \quad + \quad \begin{array}{|c|l} & \\ 3 & \text{ana} \\ & \\ 0 & \text{banana} \\ & \\ & \end{array} \quad \Longrightarrow \quad \begin{array}{|c|l} 5 & \text{a} \\ 3 & \text{ana} \\ 1 & \text{anana} \\ 0 & \text{banana} \\ 4 & \text{na} \\ 2 & \text{nana} \end{array}$$

Recursion Example



Recursion

- sort triples $S[i : i+2]$ for $i \bmod 3 \neq 0$
  (LSD-first radix sort)
- find lexicographic names $S'[1 : 2n/3]$ of triples,
  (i.e., $S'[i] < S'[j]$ iff $S[i : i+2] < S[j : j+2]$)
- $S^{12} = [S'[i] : i \bmod 3 = 1] \circ [S'[i] : i \bmod 3 = 2]$,
  suffix S^{12}_i of S^{12} represents S_{3i+1}
  suffix $S^{12}_{n/3+i}$ of S^{12} represents S_{3i+2}
- recurseOn(S^{12}) (alphabet size $\leq 2n/3$)
- Annotate the 23-suffixes with their position in rec. sol.

Sorting mod 0 Suffixes

0	c 3(h 4i h 6u 2a h 5u 1a)
1	
2	
3	h 6(u 2a h 5u 1a)
4	
5	
6	h 5(u 1a)
7	
8	

Use radix sort (LSD-order already known)

Merge SA^{12} **and** SA^{0}

$0 < 1 \Leftrightarrow c\,n < c\,n$	`4:  ( 6 )u  2 (ahua)`
$0 < 2 \Leftrightarrow cc\,n < cc\,n$	`7:  ( 5 )u  1 (a)`
	`2:  ( 4 )i  h  6 (uahua)`
`3:  h  6 u  2 (ahua)`	`1:  ( 3 )h  4 (ihuahua)`
`6:  h  5 u  1 (a)`	`5:  ( 2 )a  h  5 (ua)`
`0:  c  3 h  4 (ihuahua)`	`8:  ( 1 )a  0 0  0 (0)`

$\Downarrow$

```
8:  a
5:  ahua
0:  chihuahua
1:  hihuahua
6:  hua
3:  huahua
2:  ihuahua
7:  ua
4:  uahua
```

Analysis

1. Recursion: $T(2n/3)$ plus

   Extract triples: $O(n)$ (forall $i, i \bmod 3 \neq 0$ do . . .)

   Sort triples: $O(n)$

   (e.g., LSD-first radix sort — 3 passes)

   Lexicographic naming: $O(n)$ (scan)

   Build recursive instance: $O(n)$ (forall names do . . .)

2. $SA^0 =$sort $\{S_i : i \bmod 3 = 0\}$: $O(n)$ (1 radix sort pass)

3. merge SA^{12} and SA^0: $O(n)$ (ordinary merging with strange comparison function)

All in all: $T(n) \leq cn + T(2n/3)$

$\Rightarrow T(n) \leq 3cn = O(n)$

Implementation: Comparison Operators

```
inline bool leq(int a1, int a2,   int b1, int b2) {
  return(a1 < b1 || a1 == b1 && a2 <= b2);
}
inline bool leq(int a1, int a2, int a3,   int b1, int b2, int b3) {
  return(a1 < b1 || a1 == b1 && leq(a2,a3, b2,b3));
}
```

Implementation: Radix Sorting

```
// stably sort a[0..n-1] to b[0..n-1] with keys in 0..K from r
static void radixPass(int* a, int* b, int* r, int n, int K)
{ // count occurrences
  int* c = new int[K + 1];                              // counter array
  for (int i = 0;  i <= K;  i++) c[i] = 0;             // reset counters
  for (int i = 0;  i < n;  i++) c[r[a[i]]]++;       // count occurences
  for (int i = 0, sum = 0;  i <= K;  i++) { // exclusive prefix sums
     int t = c[i];  c[i] = sum;  sum += t;
  }
  for (int i = 0;  i < n;  i++) b[c[r[a[i]]]++] = a[i];      // sort
  delete [] c;
}
```

Implementation: Sorting Triples

```
void suffixArray(int* s, int* SA, int n, int K) {
  int n0=(n+2)/3, n1=(n+1)/3, n2=n/3, n02=n0+n2;
  int* s12  = new int[n02 + 3];  s12[n02]= s12[n02+1]= s12[n02+2]=0;
  int* SA12 = new int[n02 + 3]; SA12[n02]=SA12[n02+1]=SA12[n02+2]=0;
  int* s0   = new int[n0];
  int* SA0  = new int[n0];

  // generate positions of mod 1 and mod  2 suffixes
  // the "+(n0-n1)" adds a dummy mod 1 suffix if n%3 == 1
  for (int i=0, j=0;  i < n+(n0-n1);  i++) if (i%3 != 0) s12[j++] = i;

  // lsb radix sort the mod 1 and mod 2 triples
  radixPass(s12 , SA12, s+2, n02, K);
  radixPass(SA12, s12 , s+1, n02, K);
  radixPass(s12 , SA12, s  , n02, K);
```

Implementation: Lexicographic Naming

```
// find lexicographic names of triples
int name = 0, c0 = -1, c1 = -1, c2 = -1;
for (int i = 0;  i < n02;  i++) {
  if (s[SA12[i]] != c0 || s[SA12[i]+1] != c1 || s[SA12[i]+2] != c2) {
    name++;  c0 = s[SA12[i]];  c1 = s[SA12[i]+1];  c2 = s[SA12[i]+2];
  }
  if (SA12[i] % 3 == 1) { s12[SA12[i]/3]      = name; } // left half
  else                  { s12[SA12[i]/3 + n0] = name; } // right half
```

Implementation: Recursion

```
// recurse if names are not yet unique
if (name < n02) {
  suffixArray(s12, SA12, n02, name);
  // store unique names in s12 using the suffix array
  for (int i = 0;  i < n02;  i++) s12[SA12[i]] = i + 1;
} else // generate the suffix array of s12 directly
  for (int i = 0;  i < n02;  i++) SA12[s12[i] - 1] = i;
```

Implementation: Sorting mod 0 Suffixes

```
for (int i=0, j=0;  i < n02;  i++) if (SA12[i] < n0) s0[j++] = 3*SA12[i];
radixPass(s0, SA0, s, n0, K);
```

Implementation: Merging

```
  for (int p=0,  t=n0-n1,  k=0;  k < n;  k++) {
#define GetI() (SA12[t] < n0 ? SA12[t] * 3 + 1 : (SA12[t] - n0) * 3 + 2)
    int i = GetI(); // pos of current offset 12 suffix
    int j = SA0[p]; // pos of current offset 0  suffix
    if (SA12[t] < n0 ?
        leq(s[i],       s12[SA12[t] + n0], s[j],       s12[j/3]) :
        leq(s[i],s[i+1],s12[SA12[t]-n0+1], s[j],s[j+1],s12[j/3+n0]))
    { // suffix from SA12 is smaller
      SA[k] = i;  t++;
      if (t == n02) { // done --- only SA0 suffixes left
        for (k++;  p < n0;  p++, k++) SA[k] = SA0[p];
      }
    } else {
      SA[k] = j;  p++;
      if (p == n0)  { // done --- only SA12 suffixes left
        for (k++;  t < n02;  t++, k++) SA[k] = GetI();
      }
    }
  }
  delete [] s12; delete [] SA12; delete [] SA0; delete [] s0;
}
```

Tuning

- Eliminate mod, div
- MSD-first radix sort
- Use partial sorting of triples embracing recursion level
- Various locality improvements

Bottom line: Beats previous algorithms for difficult inputs. (but still $\approx 2\times$ slower for easy inputs.)

External Memory Implementation

Recursion: $T(2n/3)$

Extract triples: scan input

Sort triples: sort (once)

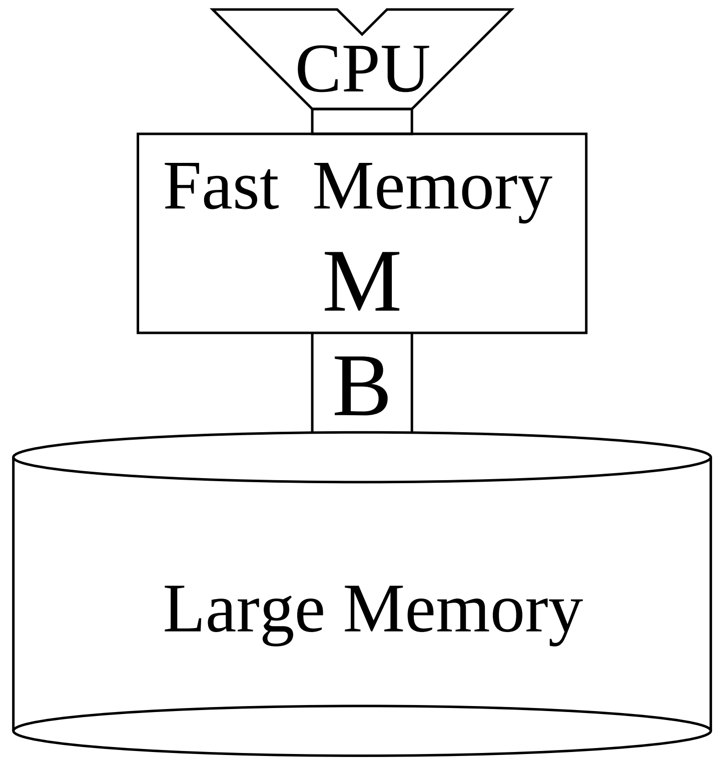
Lexicographic naming:
scan sorted triples

Build recursive instance: sort

Annotate input: sort

Sort the rest: sort(once)

All in all: $O(T_{\mathrm{sort}}(n))$ I/Os



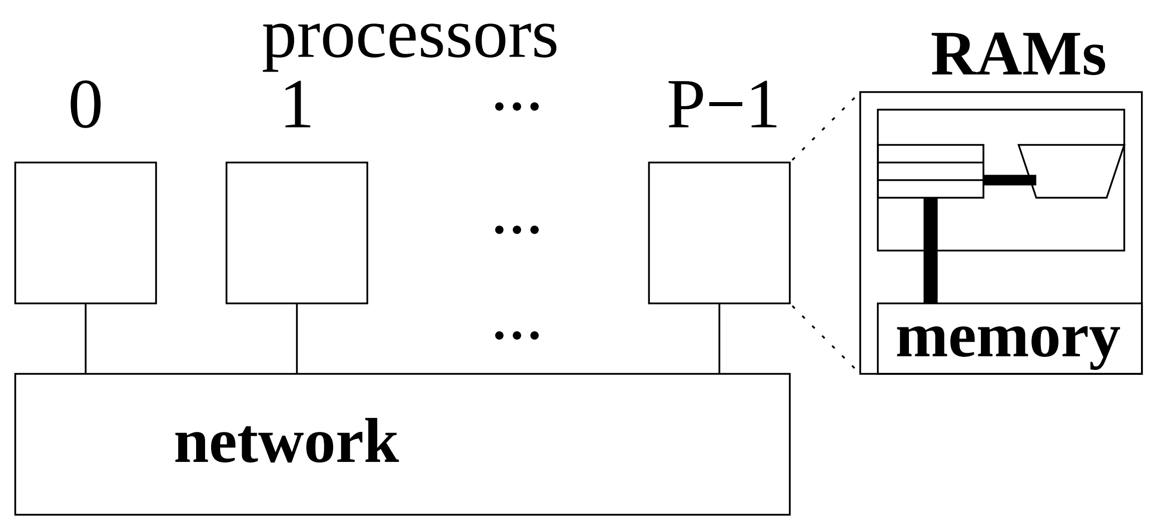
Parallel Implementation

sorting: parallel integer sorting

lexicographic naming: prefix sums $O(n/P + \log P)$

Integer alphabet: $O(n^{\varepsilon})$ time, $O(n)$ work

Comparison based: $O(\log^2 n)$ time, $O(n \log n)$ work



More Linear Time Algorithms

- [Kim Sim Park Park CPM 03]
  A direct implementation of Farach's idea. Complicated.
- [Ko Aluru CPM 03]
  a different recursion. Still somewhat complicated
- [Kärkkäinen Burkhardt CPM 03]
  Cycle covers allow generalization to smaller recursive subproblems. Extra space $O(\varepsilon n)$. Linear time with additional ideas from here.
- [Hong Sadakane Sung FOCS 04]
  extra space $O(n)$ bits. Farach's idea again.

None looks easy to parallelize/externalize

Suffix Array Construction: Conclusion

- simple, direct, linear time suffix array construction
- easy to adapt to advanced models of computation
- generalization to cycle covers yields space efficient implementation

Future/Ongoing Work

- Implementation (internal/external/parallel)
- Large scale applications