Priority Queues

Maintain set M of Elements with Keys

Insert:

DeleteMin:

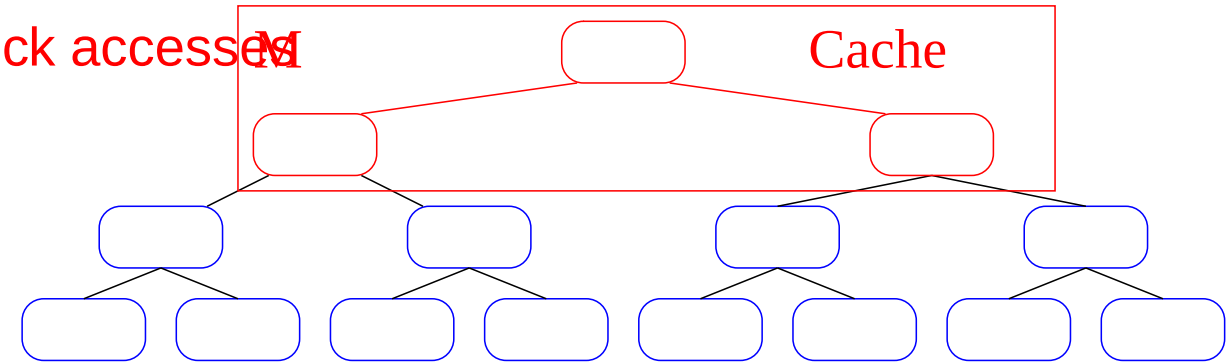
Applications (no additional operations):

- Multi-way Merging (small)
- Greedy Algorithms (e.g., machine scheduling) (small–medium)
- Discrete event simulation (medium–large)
- Branch-and-Bound (large)
- Building long runs (large)
- Time forward processing (huge)

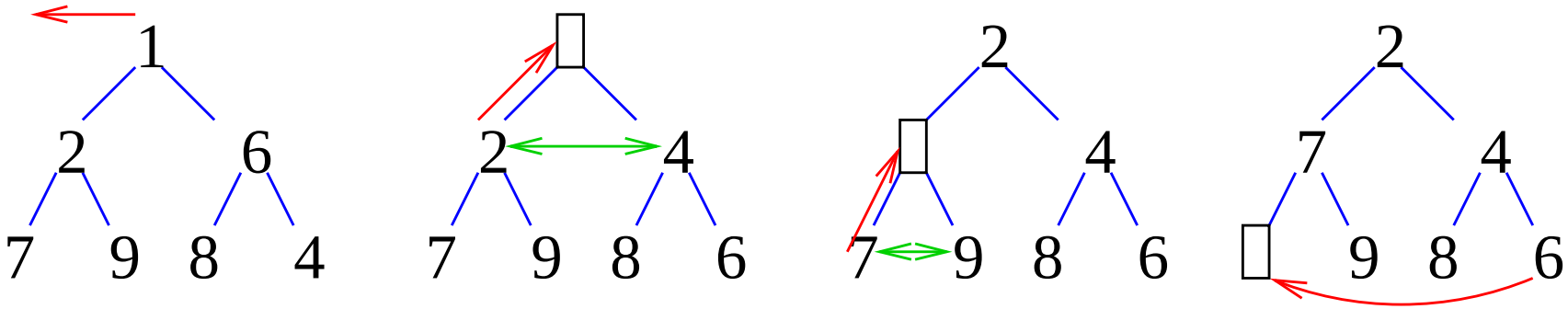
Binary Heaps

best comparison based “flat memory” algorithm

+ On average constant time for insertion
+ On average $\log n + O(1)$ key comparisons per delete-Min using the “bottom-up” heuristics [Wegener 93].
− $\approx 2\log(n/M)$ block accesses per delete-Min



Bottom Up Heuristics



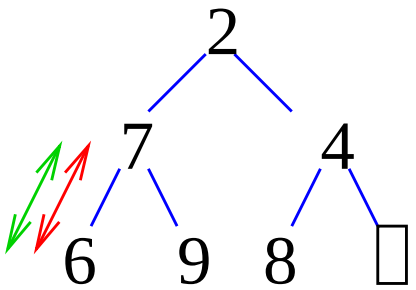
delete Min

O(1)

sift down hole

log(n)

compare swap move



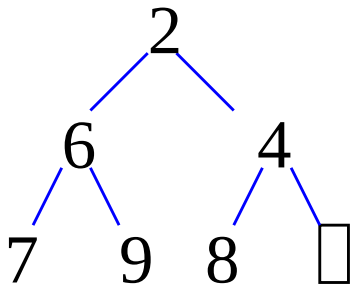
Factor two faster

than naive implementation

sift up

O(1)

average



Der Wettbewerber fit gemacht:

```
int i=1, m=2, t = a[1];
m += (m != n  &&  a[m] > a[m + 1]);
if (t > a[m]) {
  do { a[i] = a[m];
       i    = m;
       m    = 2*i;
       if (m > n) break;
       m += (m != n  &&  a[m] > a[m + 1]);
  } while (t > a[m]);
  a[i] = t;}
```

Keine signifikanten Leistungsunterschiede auf meiner Maschine

(heapsort von random integers)

Vergleich

Speicherzugriffe: $o(1)$ weniger als top down $o(\log n)$ worst case. bei effizienter Implementierung

Elementvergleiche: $\approx \log n$ weniger für bottom up (average case) aber die sind leicht vorhersagbar

Aufgabe: siftDown mit worst case $\log n + o(\log \log n)$ Elementvergleichen

Heapkonstruktion

Procedure buildHeapBackwards

for $i := \lfloor n/2 \rfloor$ **downto** 1 **do** siftDown(i)

Procedure buildHeapRecursive$(i : \mathbb{N})$

if $4i \leq n$ **then**

buildHeapRecursive$(2i)$

buildHeapRecursive$(2i+1)$

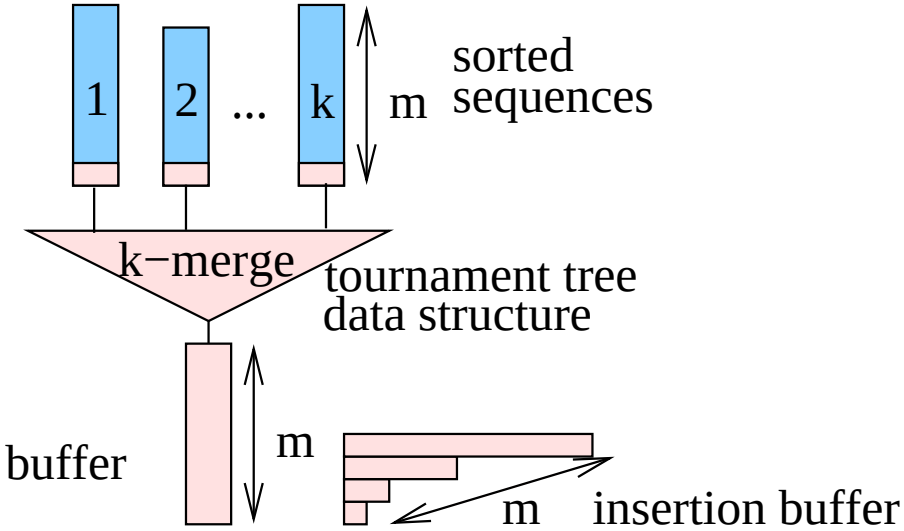
siftDown(i)

Rekursive Funktion für große Eingaben $2\times$ schneller!

(Rekursion abrollen für 2 unterste Ebenen)

Aufgabe: Erklärung

Medium Size Queues ($km \ll M^2/B$ Insertions)



Insert: Initially into insertion buffer.

Overflow $\longrightarrow$

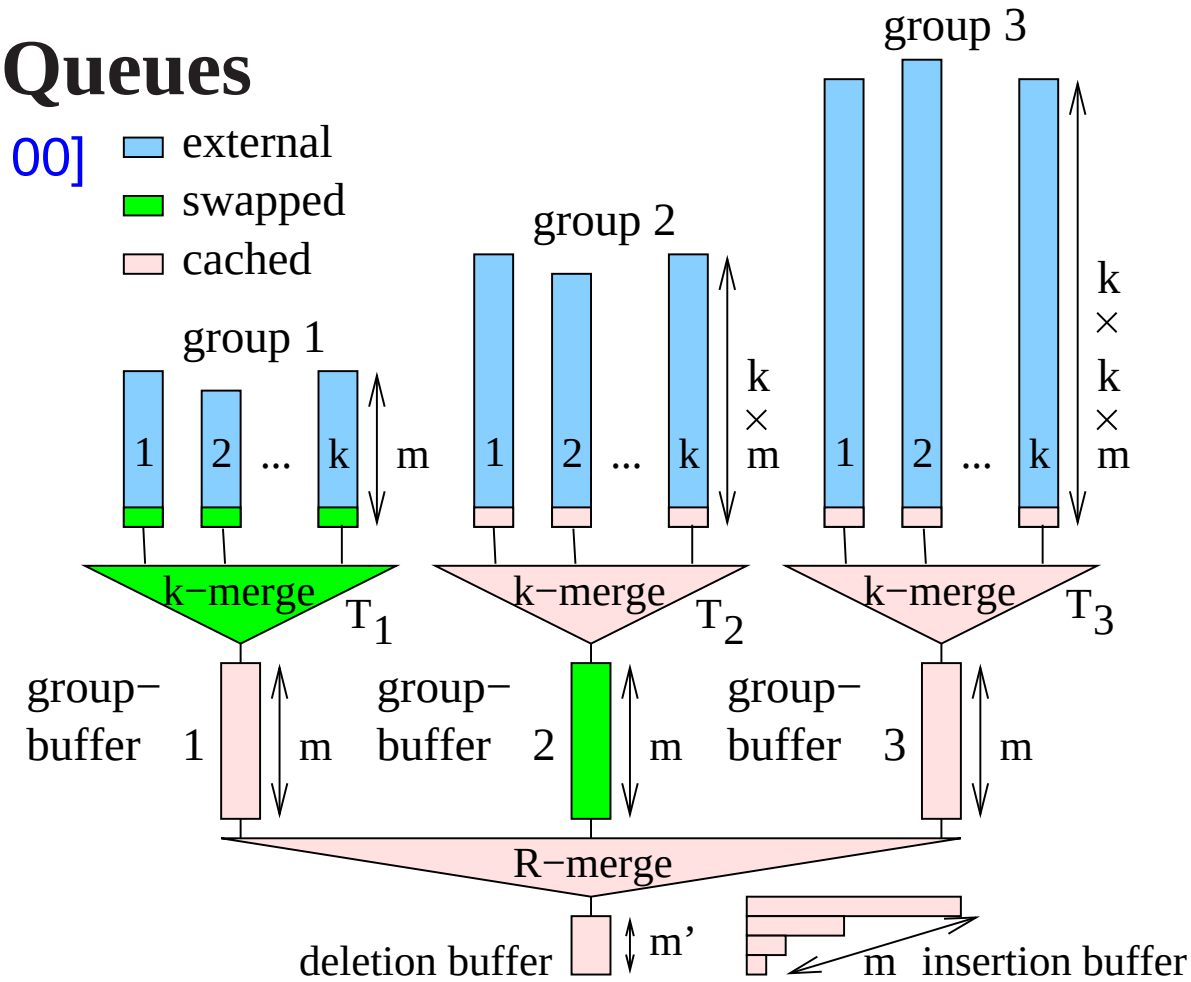
sort; merge with deletion buffer; write out largest elements.

Delete-Min: Take minimum of insertion and deletion buffer.

Refill deletion buffer if necessary.

Large Queues

[Sanders 00]



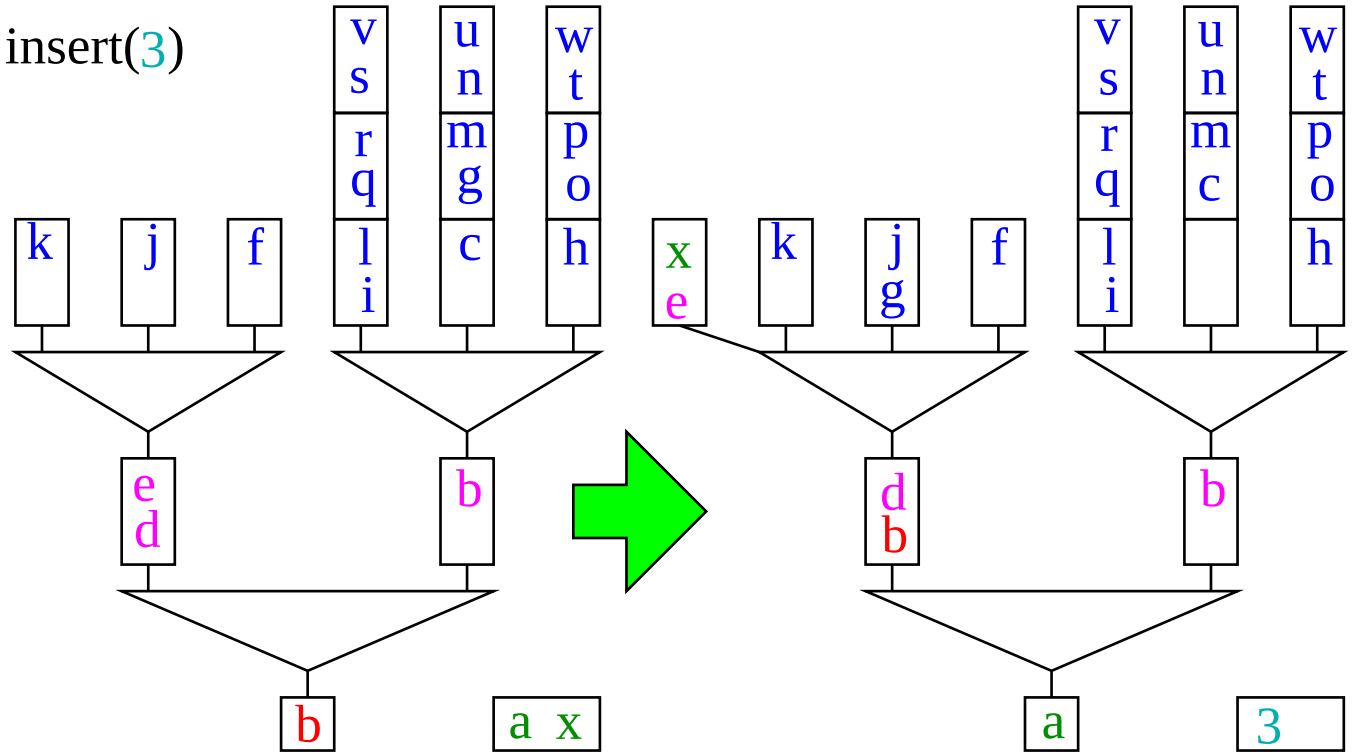
insert: group full $\longrightarrow$ merge group; shift into next group.

merge invalid group buffers and move them into group 1.

Delete-Min: Refill. $m' \ll m$. nothing else

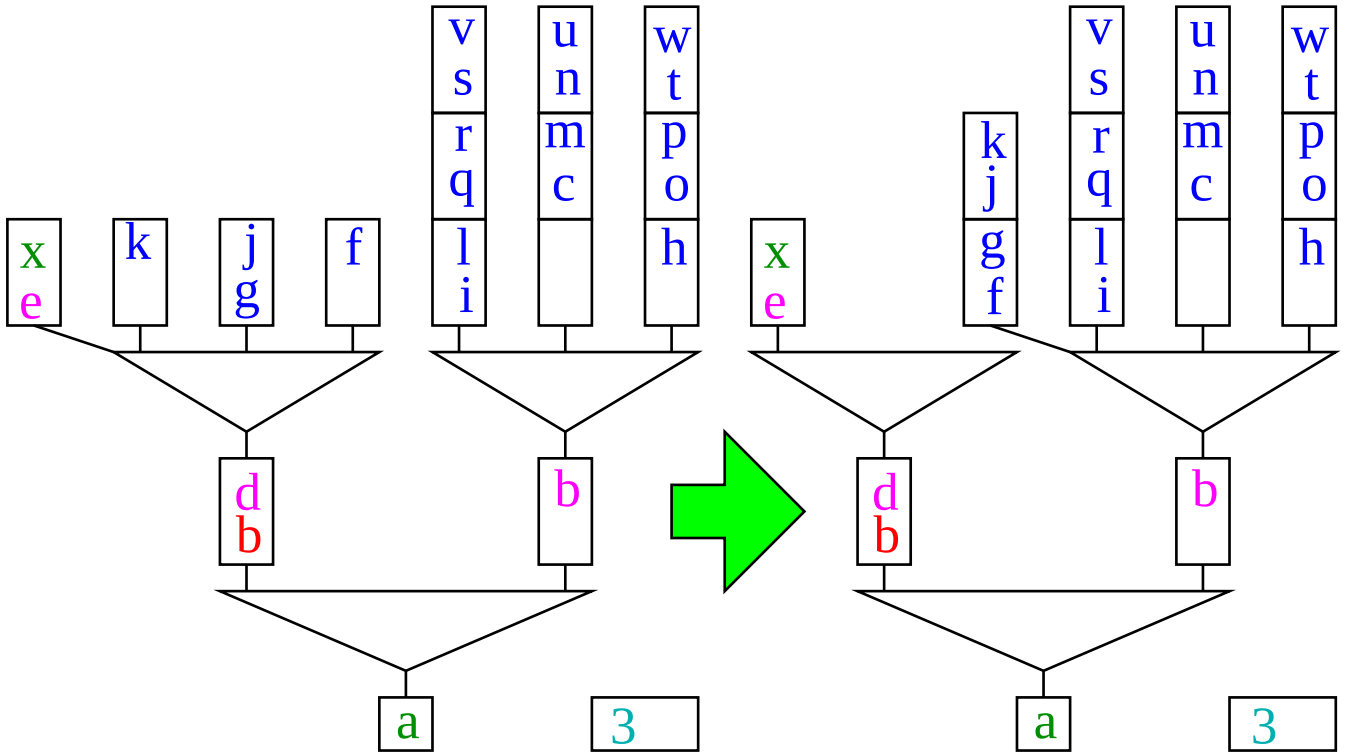
Example

Merge insertion buffer, deletion buffer, and leftmost group buffer



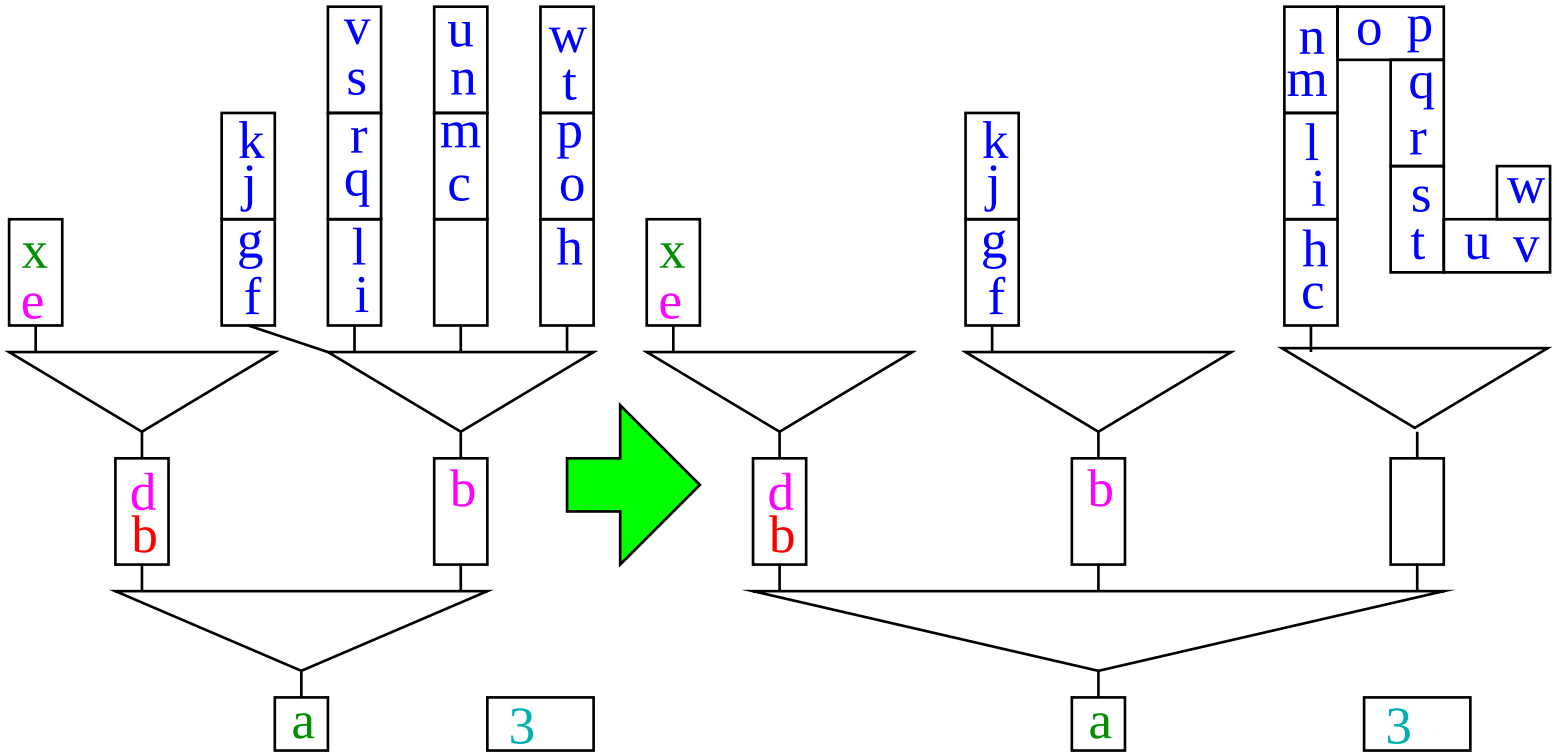
Example

Merge group 1



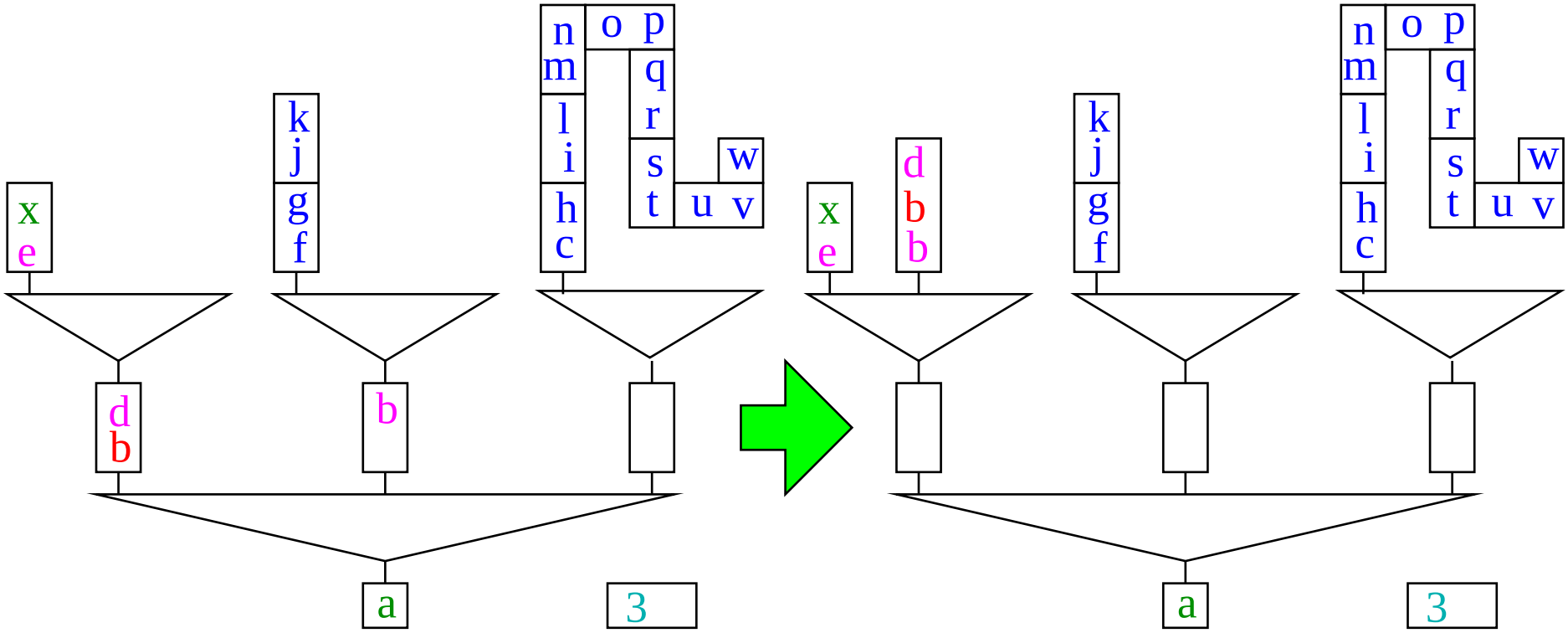
Example

Merge group 2



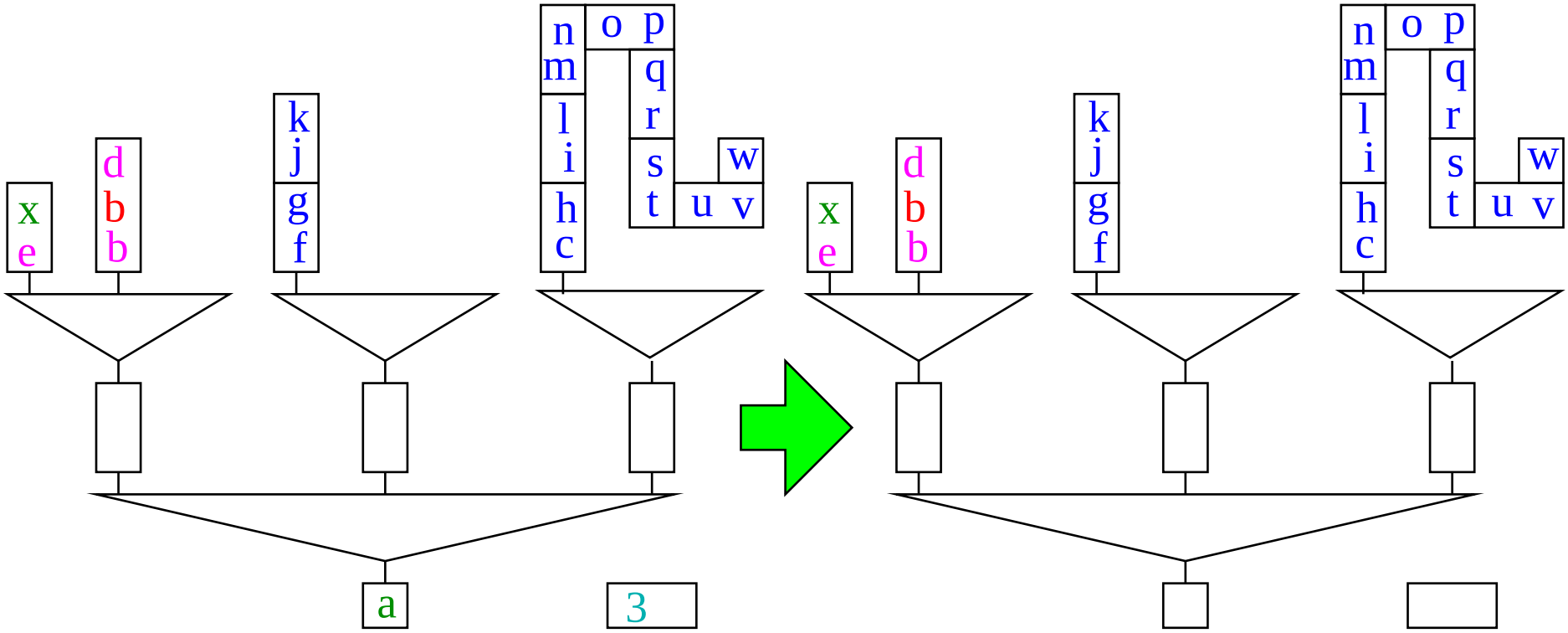
Example

Merge group buffers



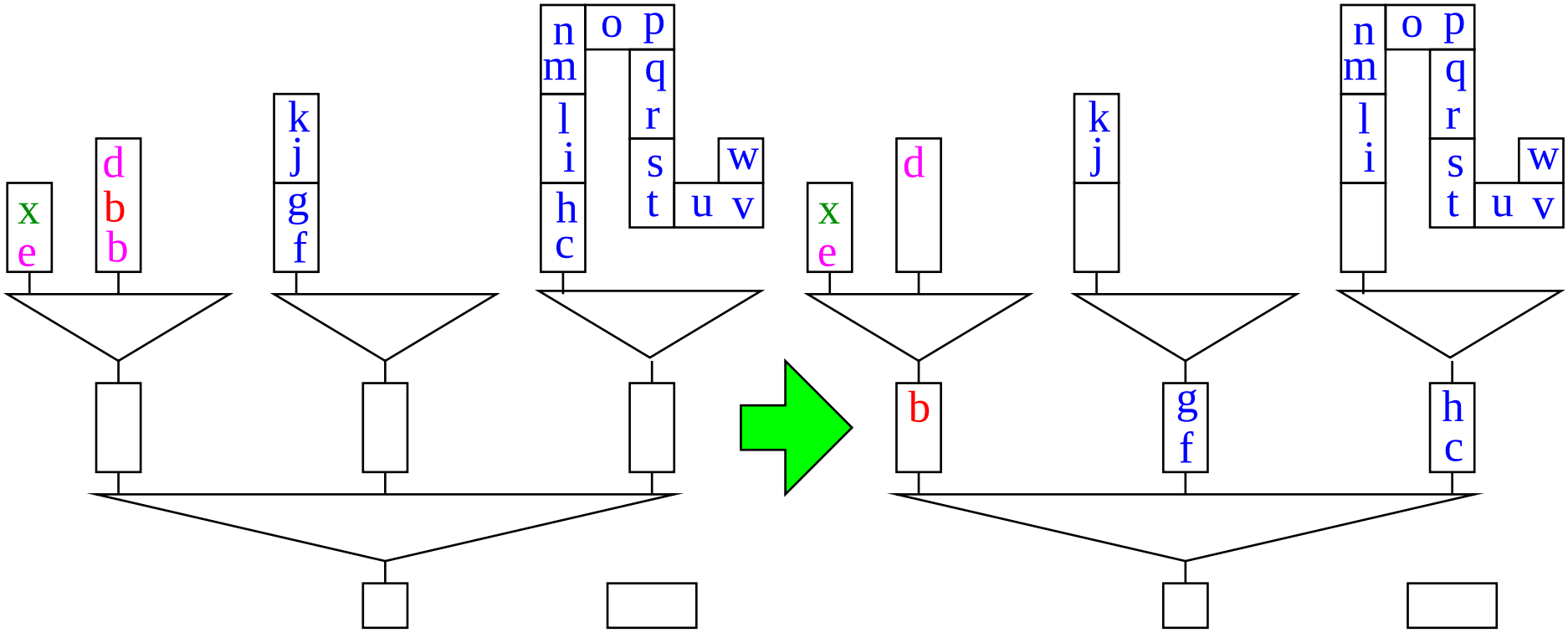
Example

DeleteMin $\rightsquigarrow$ 3; DeleteMin $\rightsquigarrow$ a;



Example

DeleteMin $\rightsquigarrow$ b



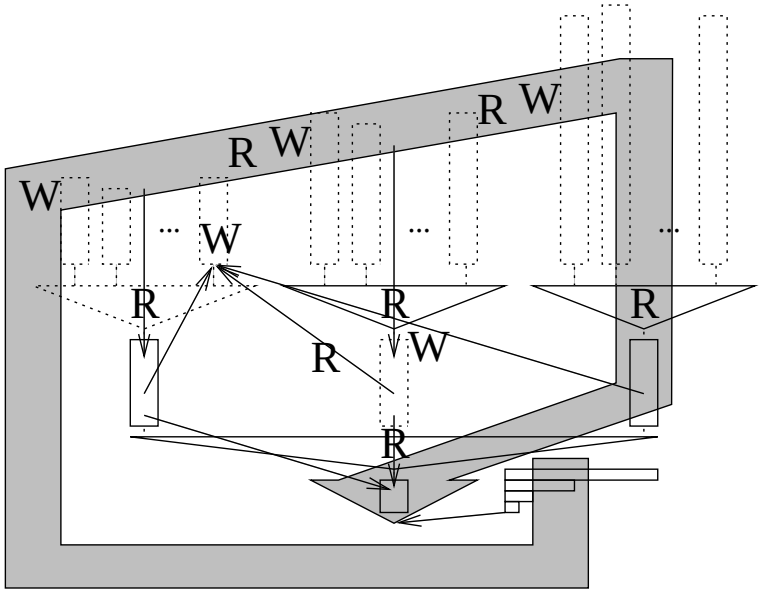
Analysis

- I insertions, buffer sizes $m = \Theta(M)$
- merging degree $k = \Theta(M/B)$

block accesses: $\text{sort}(I)$ + "small terms"

key comparisons: $I \log I$ + "small terms"

(on average)



Other (similar, earlier) [Arge 95, Brodal-Katajainen 98, Brengel et al. 99, Fadel et al. 97] data structures spend a factor ≥ 3 more I/Os to replace I by queue size.

Implementation Details

- Fast routines for 2–4 way merging keeping smallest elements in registers
- Use sentinels to avoid special case treatments (empty sequences, ...)
- Currently heap sort for sorting the insertion buffer
- $k \neq M/B$: multiple levels, limited associativity, TLB

Experiments

Keys: random 32 bit integers

Associated information: 32 dummy bits

Deletion buffer size: 32

Group buffer size: 256

Merging degree k: 128

Near optimal

: performance on

all machines tried!

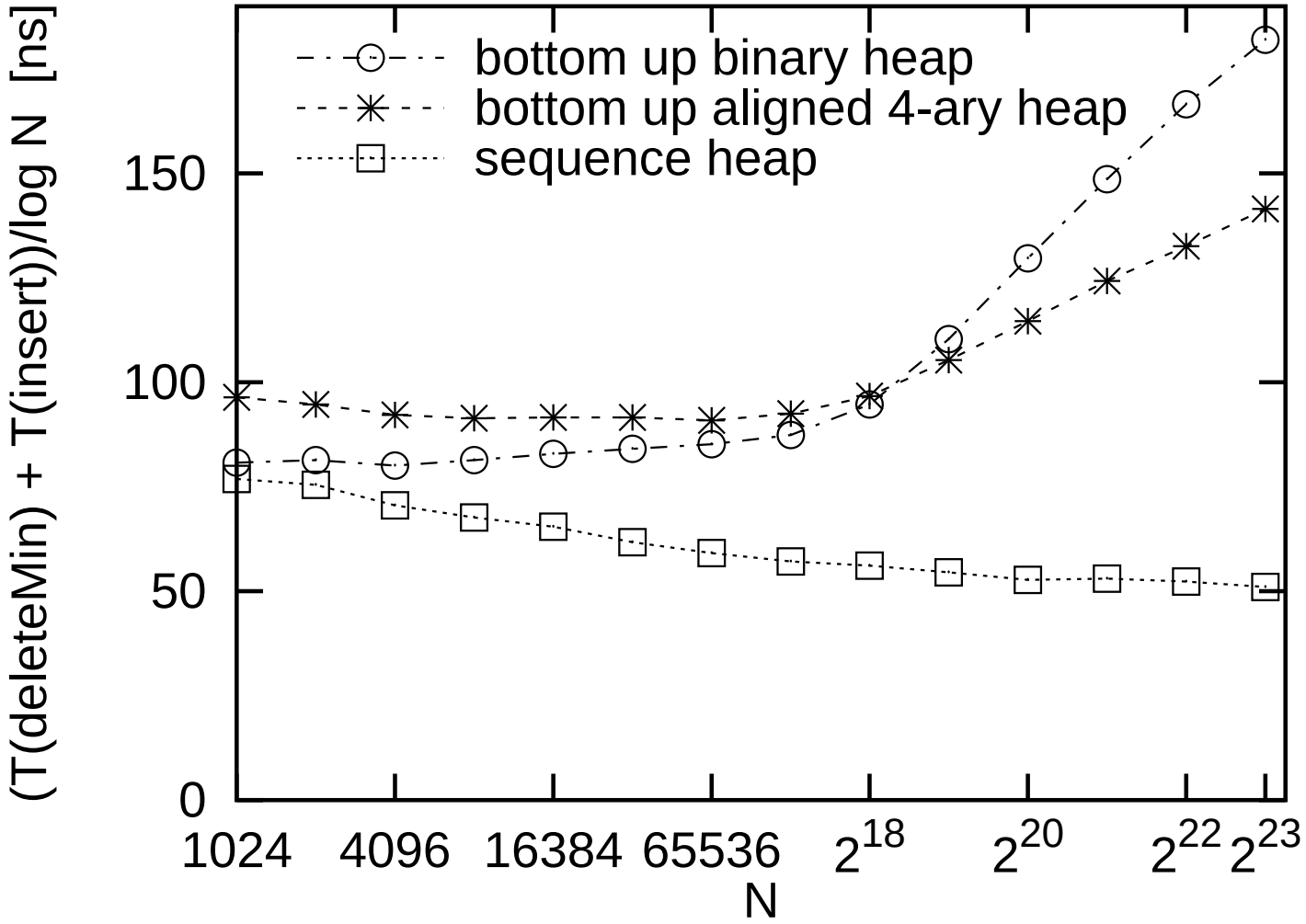
Compiler flags: Highly optimizing, nothing advanced

Operation Sequence:

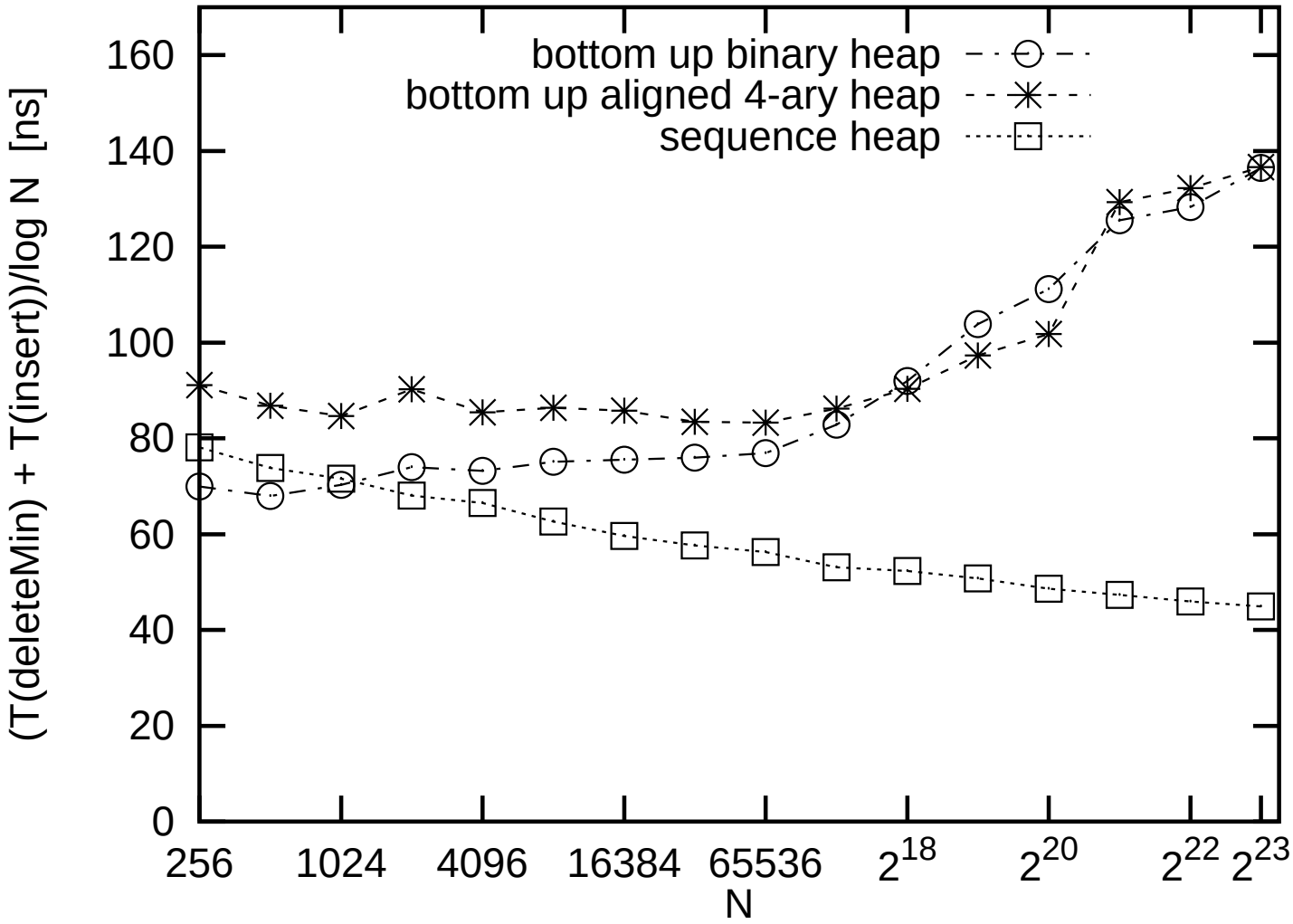
$$\left(\text{Insert-DeleteMin-Insert}\right)^N \left(\text{DeleteMin-Insert-DeleteMin}\right)^N$$

Near optimal performance on all machines tried!

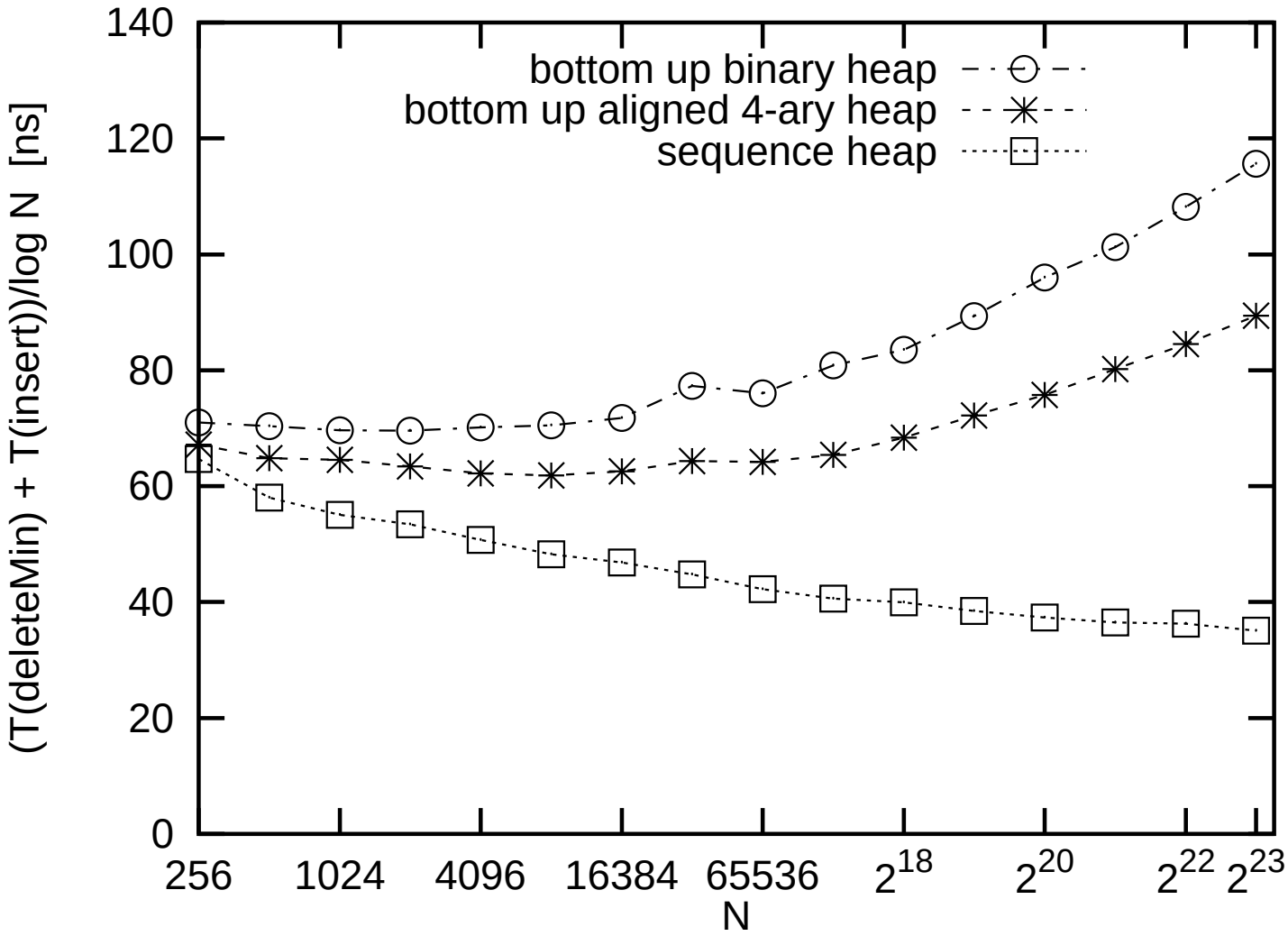
MIPS R10000, 180 MHz



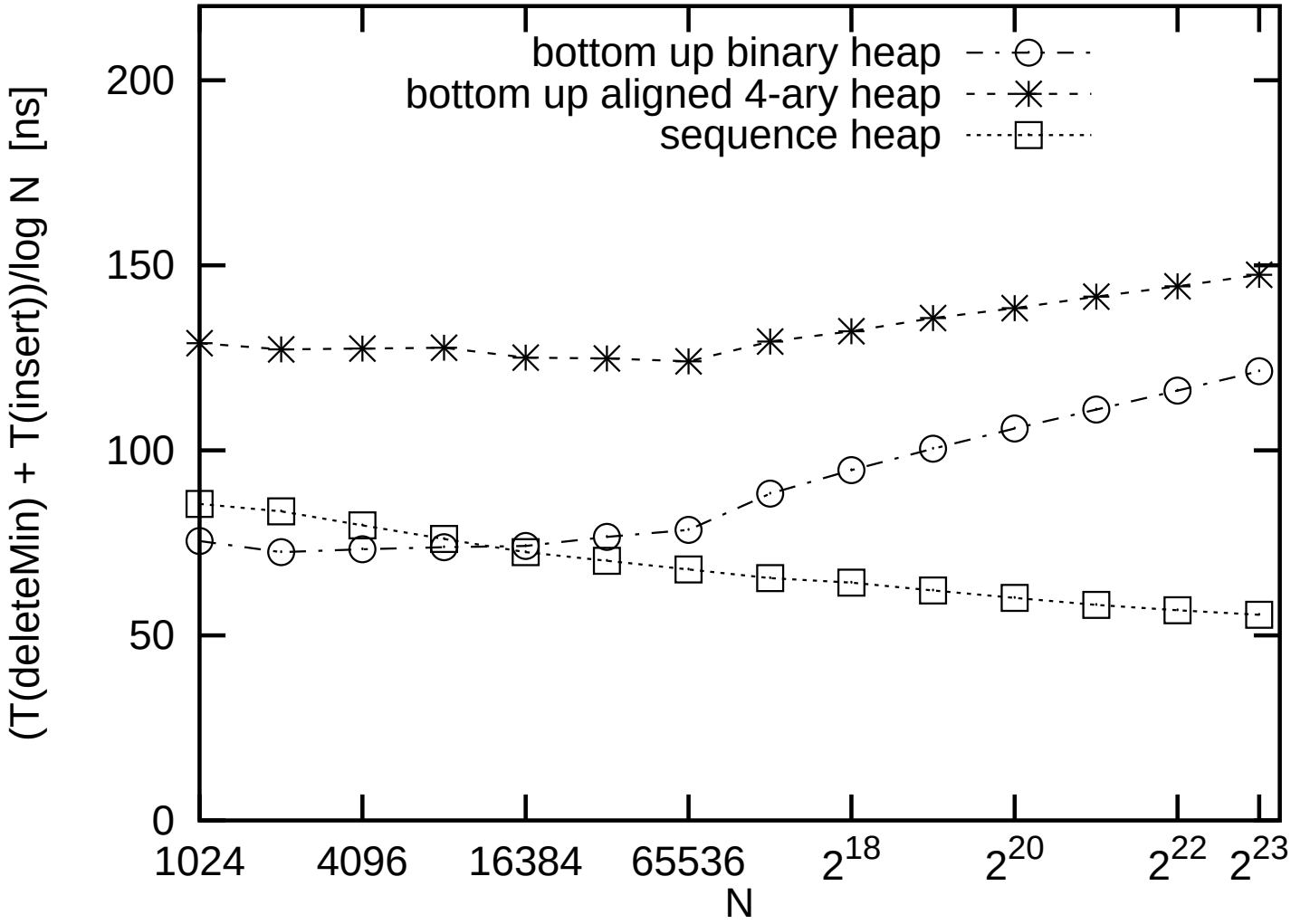
Ultra-SparcIIi, 300 MHz



Alpha-21164, 533 MHz

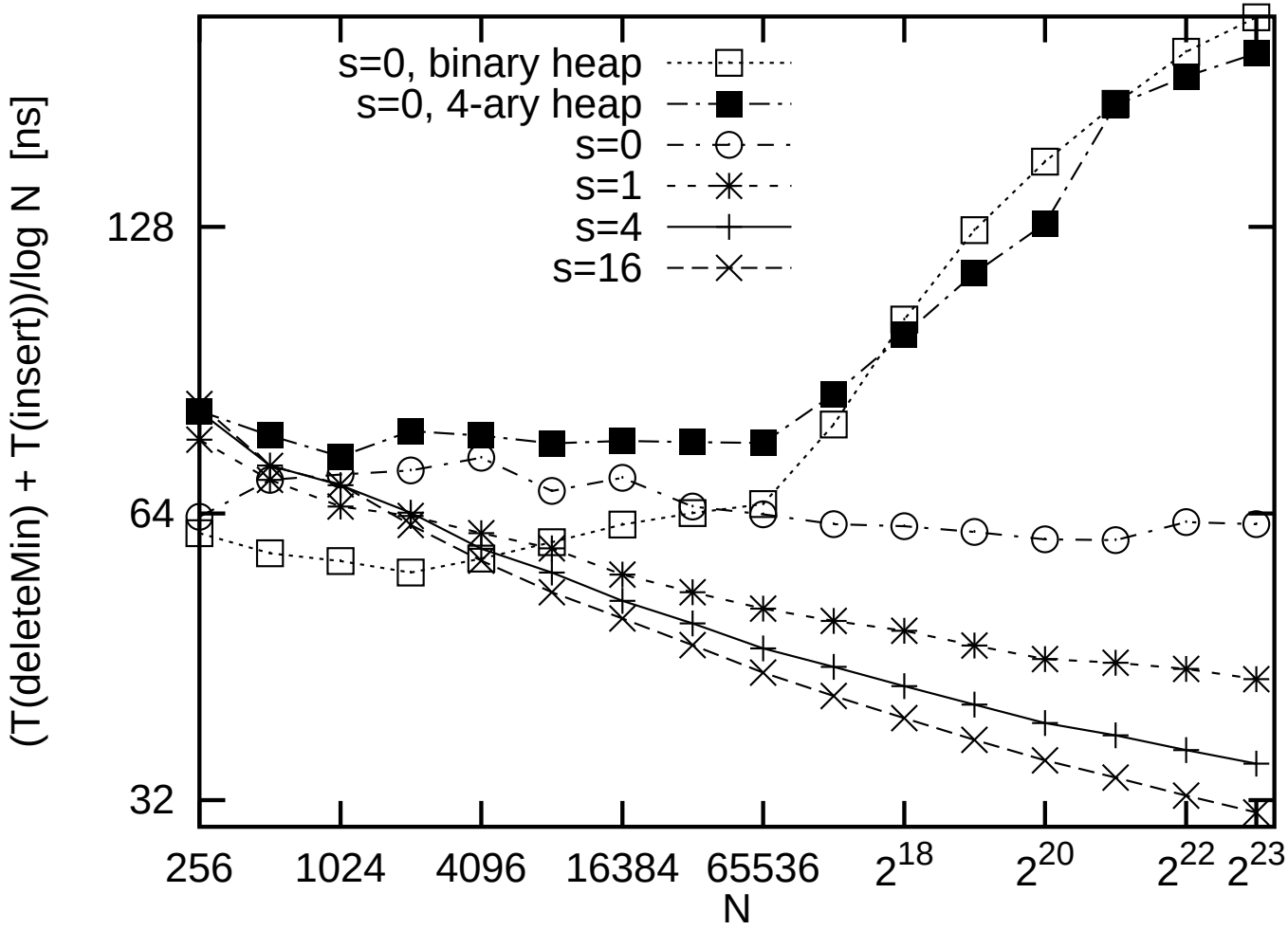


Pentium II, 300 MHz



$(\text{insert}\ (\text{deleteMin insert})^s)^N$

$(\text{deleteMin}\ (\text{insert deleteMin})^s)^N$



Methodological Lessons

If you want to compare small constant factors in execution time:

- Reproducability demands publication of source codes
  (4-ary heaps, old study in Pascal)
- Highly tuned codes in particular for the competitors
  (binary heaps have factor 2 between good and naive implementation).
  How do you compare two mediocre implementations?
- Careful choice/description of inputs
- Use multiple different hardware platforms
- Augment with theory (e.g., comparisons, data dependencies, cache faults, locality effects . . .)

Open Problems

- Integrate into STL
- Dependence on size rather than number of insertions
- Parallel disks
- Space efficient implementation
- Multi-level cache aware or cache-oblivious variants

1 Adressable Priority Queues

Procedure build$(\{e_1,\ldots,e_n\})$ $M := \{e_1,\ldots,e_n\}$

Function size **return** $|M|$

Procedure insert(e) $M := M \cup \{e\}$

Function min **return** $\min M$

Function deleteMin $e := \min M$; $M := M \setminus \{e\}$; **return** e

Function remove$(h : \text{Handle})$ $e := h$; $M := M \setminus \{e\}$; **return** e

Procedure decreaseKey$(h : \text{Handle}, k : \text{Key})$ **assert** key$(h) \geq k$; key$(h) := k$

Procedure merge(M') $M := M \cup M'$

Quick-Heaps

Idea:

Partition M into subsets $M_1,\ldots,M_k$

Invariant:

$\forall 1 \leq i < j \leq k, e_i \in M_i, e_j \in M_j : e_i \leq e_j$

$|M_i|$ is “approximately” growing geometrically with i

Quick-Heaps – Insert

Procedure insert(e)

for $i := k$ **downto** 1 **while** $e < \min M_i$ **do** ;// define $\min M_0 = -\infty$

if $i = 0$ **then** i:= 1; $\min M_1 := e$

M_i:= $M_i \cup \{e\}$

decreaseKey: similar

Quick-Heaps – deleteMin

Procedure deleteMin

$e := \min M_1$

$M_1 := M_1 \setminus \{e\}$

if $M_1 = \emptyset$ **then**

(recursively) partition M_2 and renumber

to reestablish invariant

return e

Quick-Heaps – Analysis

Paredes: average case complexity $O(\log n)$

Experiments: with very favorable constant factors.

Open Problem:

Make fit for worst case inputs.

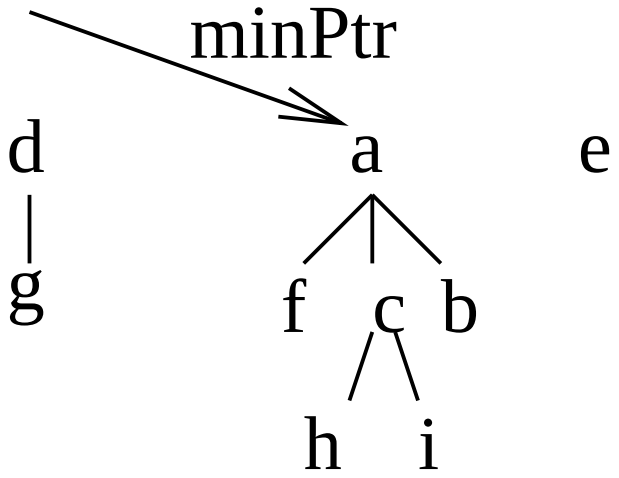
$O(1)$ expected cost for decreaseKey?

Idea: active rebalancing

Heap Ordered Forest

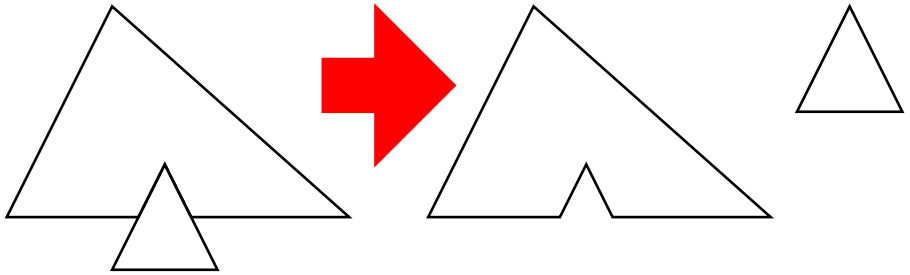
Basic Data Structure

A forest of heap-ordered trees

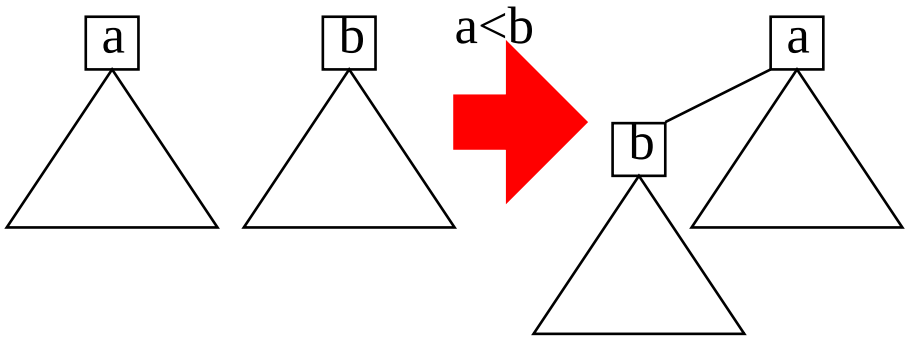


Manipulating Forests

Cut:



Link:



Pairing Heaps

Procedure insertItem$(h$: Handle$)$

newTree(h)

Procedure newTree$(h$: Handle$)$

forest:= forest$\cup\{h\}$

if $e <$ min **then** minPtr:= h

Pairing Heaps

Procedure decreaseKey$(h$: Handle$, k$: Key$)$

key$(h):= k$

if h is not a root **then** cut(h)

Pairing Heaps

Function deleteMin : Handle

m:= minPtr

forest:= forest $\setminus \{m\}$

foreach child h of m **do** newTree(h)

Perform a pairwise link of the tree roots in forest

return m

Pairing Heaps

Procedure merge$(o$: AdressablePQ$)$

if minPtr $>$ o.minPtr **then** minPtr:= o.minPtr

forest:= forest $\cup$ o.forest

o.forest:= $\emptyset$

Fibonacci Heaps (A sample from the Zoo)

Ranks: initially zero, increases for root of a link

Union by rank: Only link roots of equal rank

Mark nodes that lost a child

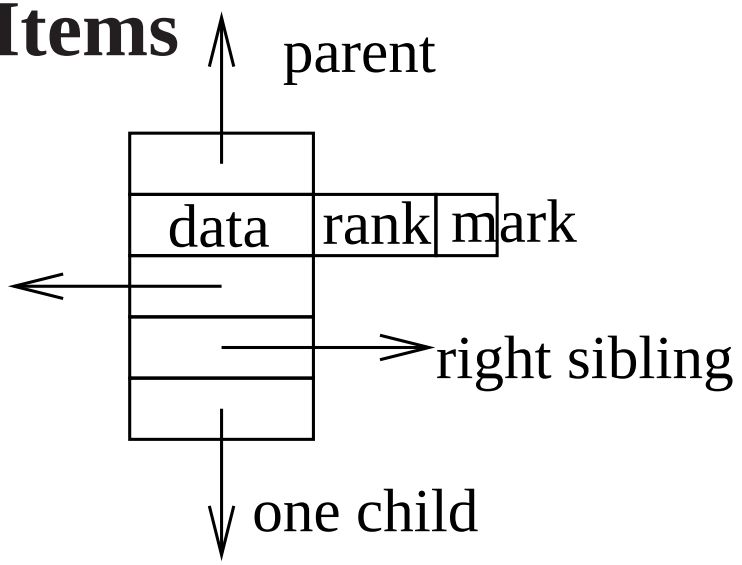
Cascading cuts: cut marked nodes (i.e., lost two childs)

Amortized complexity: $O(\log n)$ for deleteMin

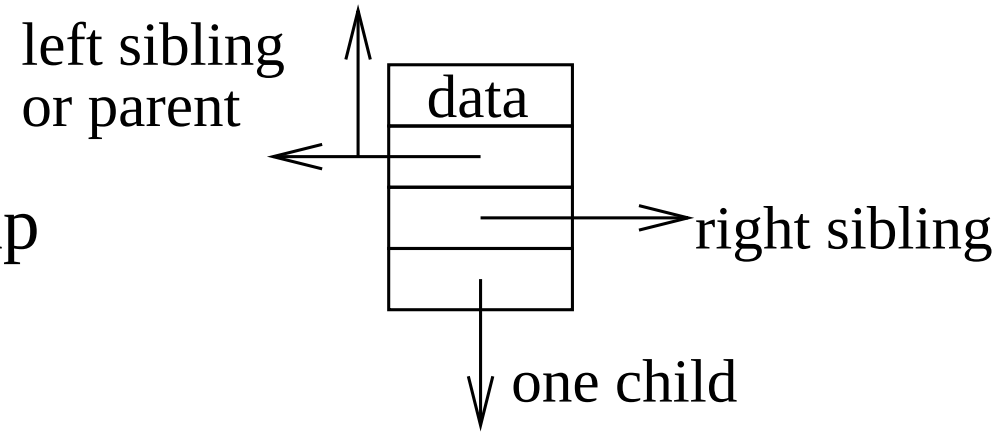
$O(1)$ for all other operations

Representing the Tree Items

Fibonacci Heap



Pairing Heap

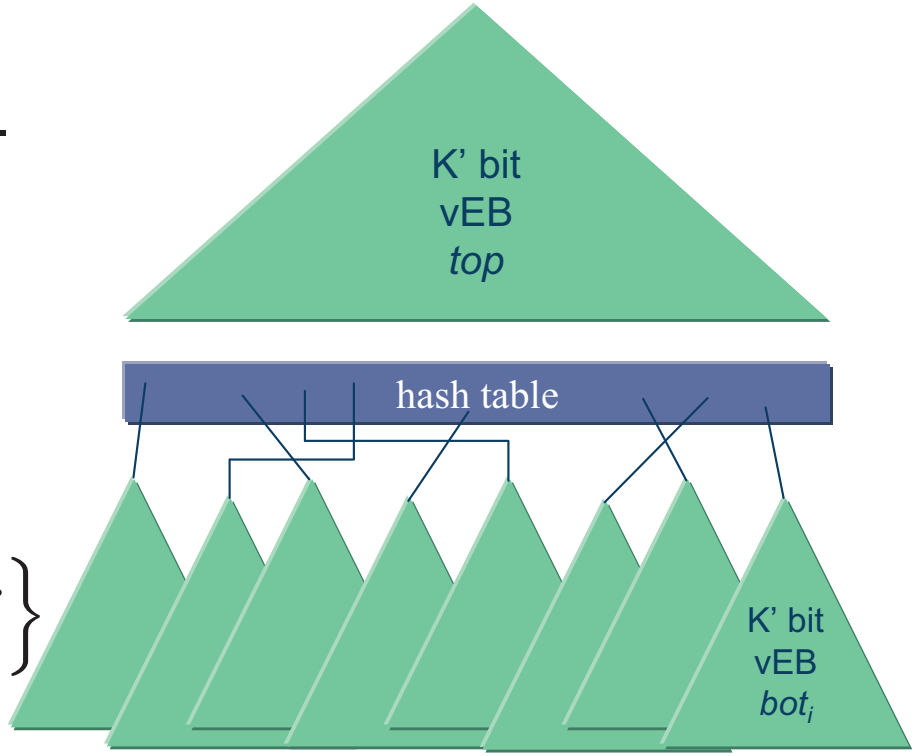


Addressable Priority Queues: Todo

- No recent comparison of efficient implementations
- No tight analysis of pairing heaps
- No implementation of compromises, e.g., thin heaps [Kaplan Tarjan 99] (three pointers, no mark, slightly more complicated balancing)
- No implementation of worst case efficient variants
- Study implementation tricks: two pointers per item? sentinels,. . .
- (Almost) nothing known for memory hierarchies

2 van Emde-Boas Search Trees

- Store set M of $K = 2^k$-bit integers.
  later: associated information
- $K = 1$ or $|M| = 1$: store directly
- $K' := K/2$
- $M_i := \left\{ x \bmod 2^{K'} : x \textbf{ div } 2^{K'} = i \right\}$
- root points to nonempty M_i-s
- top $t = \{ i : M_i \neq \emptyset \}$
- insert, delete, search in $O(\log K)$ time



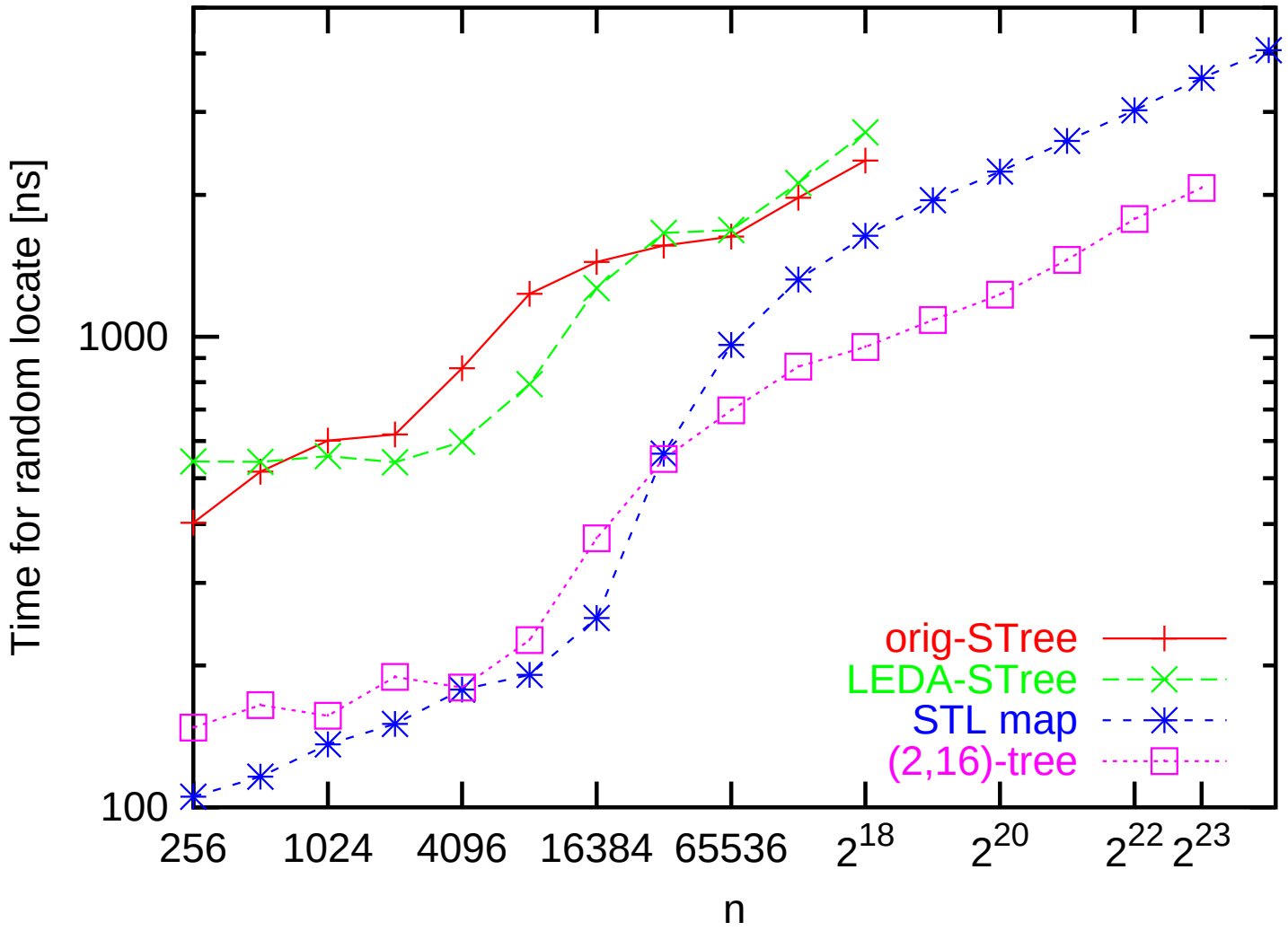
Comparison with Comparison Based Search Trees

Ideally: $\log n \rightsquigarrow \log\log n$

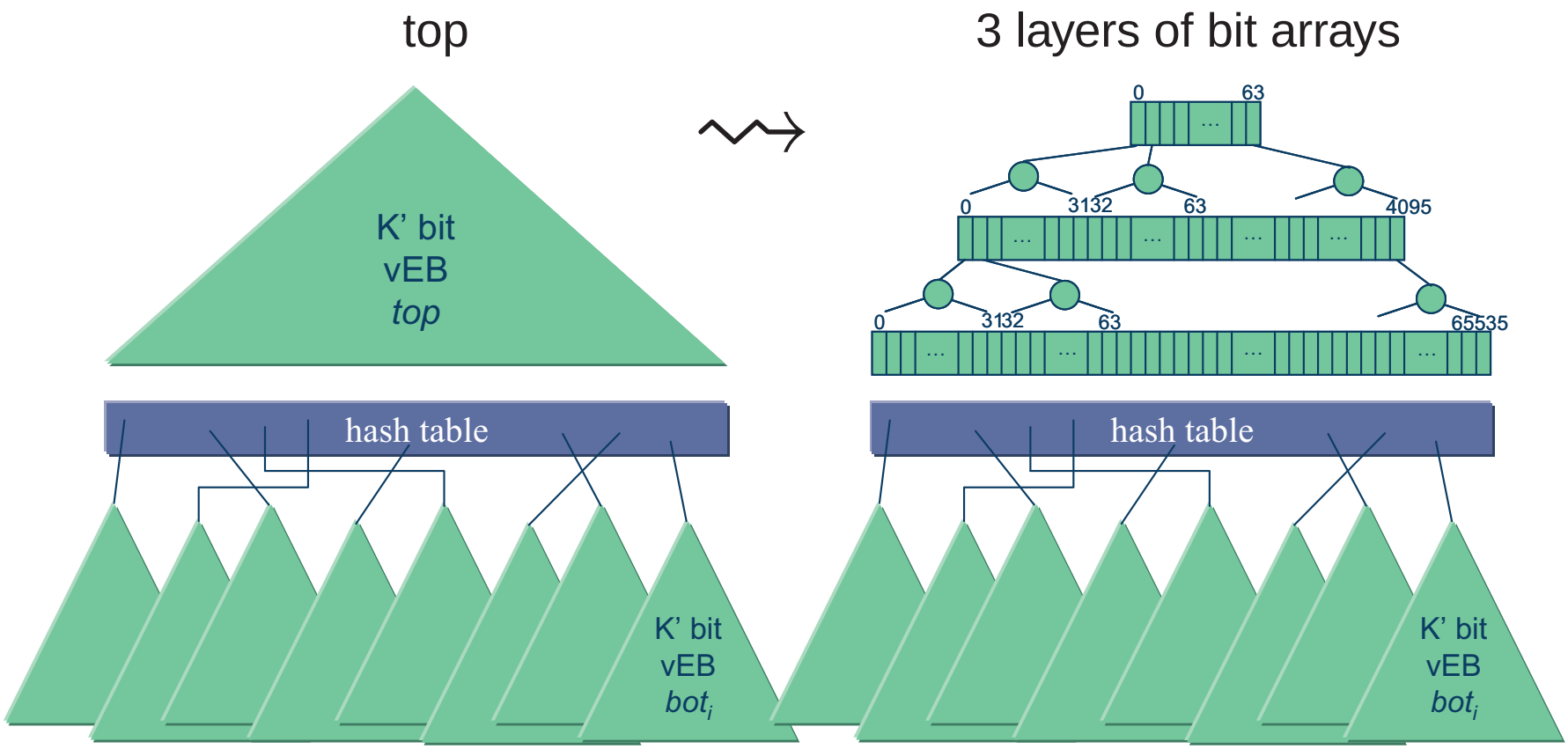
Problems:

Many special case tests

High space consumption



Efficient 32 bit Implementation



Layers of Bit Arrays

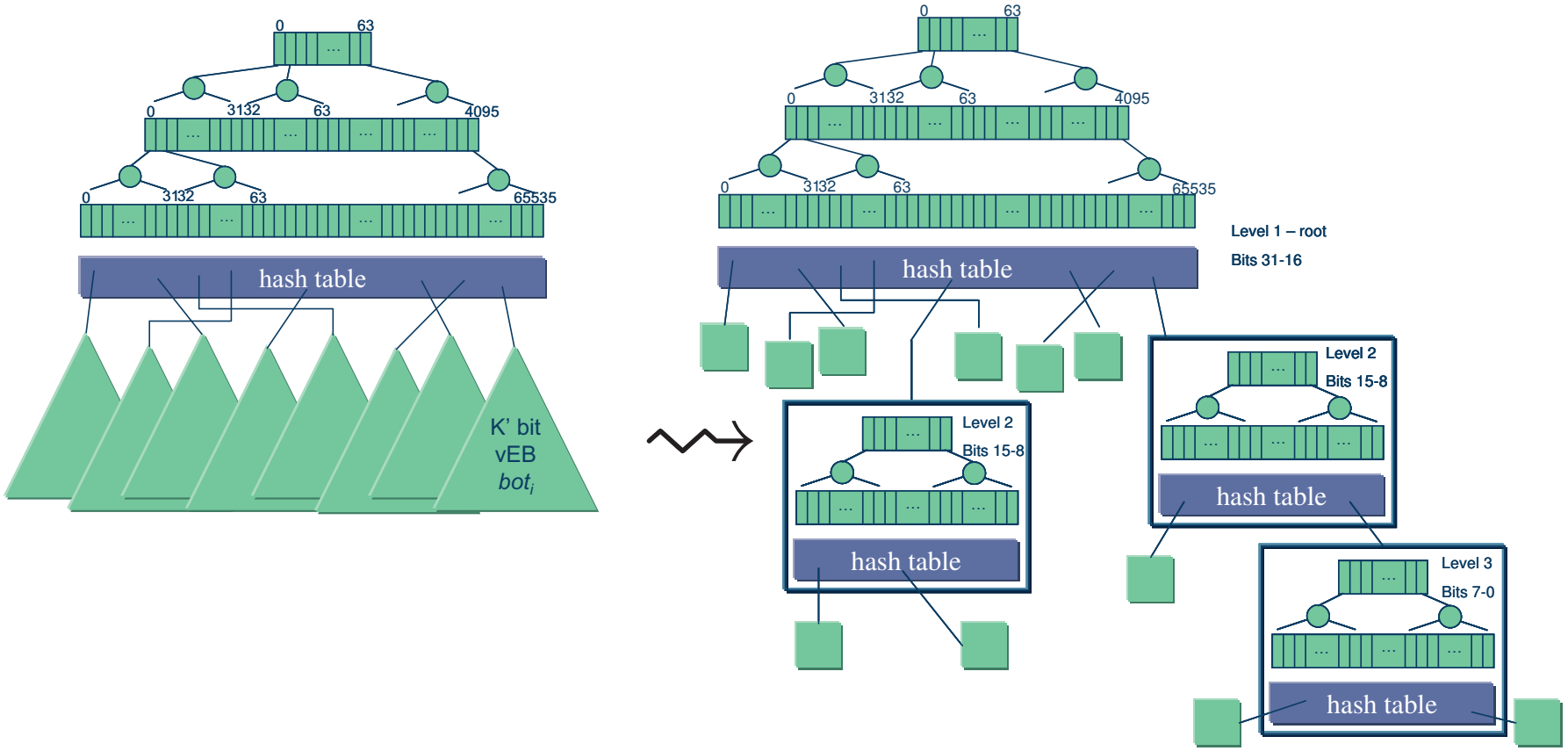
$t^1[i] = 1$ iff $M_i \neq \emptyset$

$t^2[i] = t^1[32i] \vee t^1[32i+1] \vee \cdots \vee t^1[32i+31]$

$t^3[i] = t^2[32i] \vee t^2[32i+1] \vee \cdots \vee t^2[32i+31]$

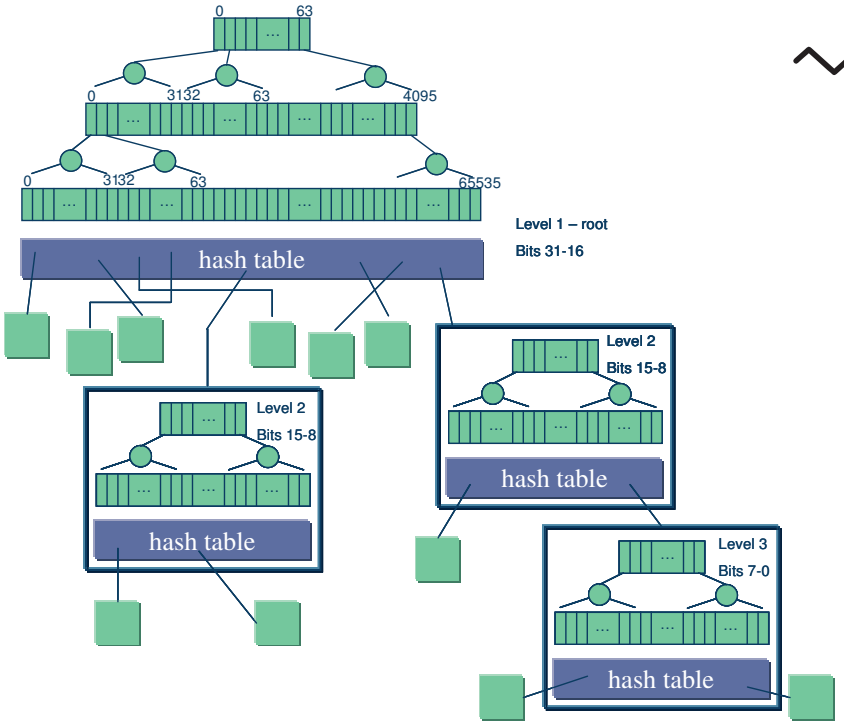
Efficient 32 bit Implementation

Break recursion after 3 layers

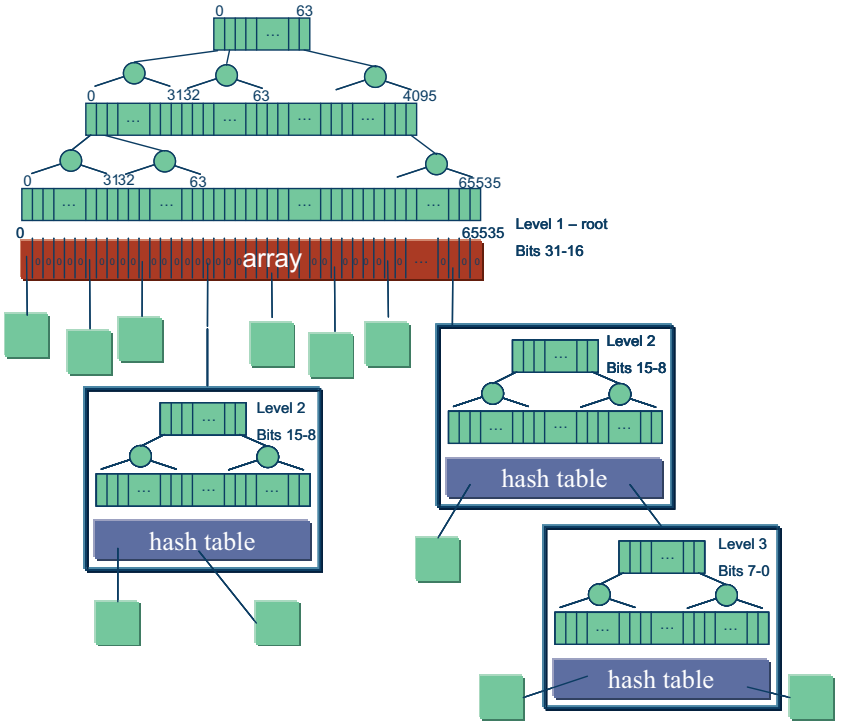


Efficient 32 bit Implementation

root hash table

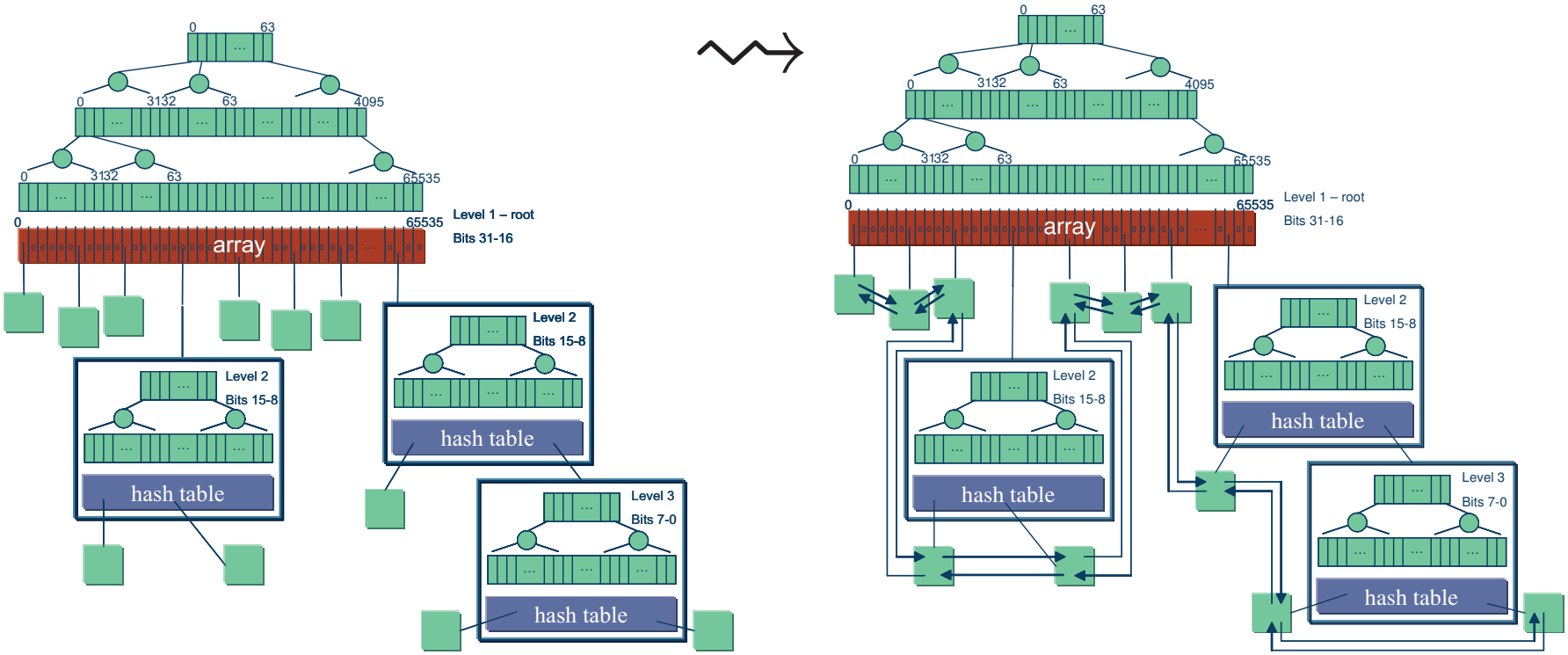


root array

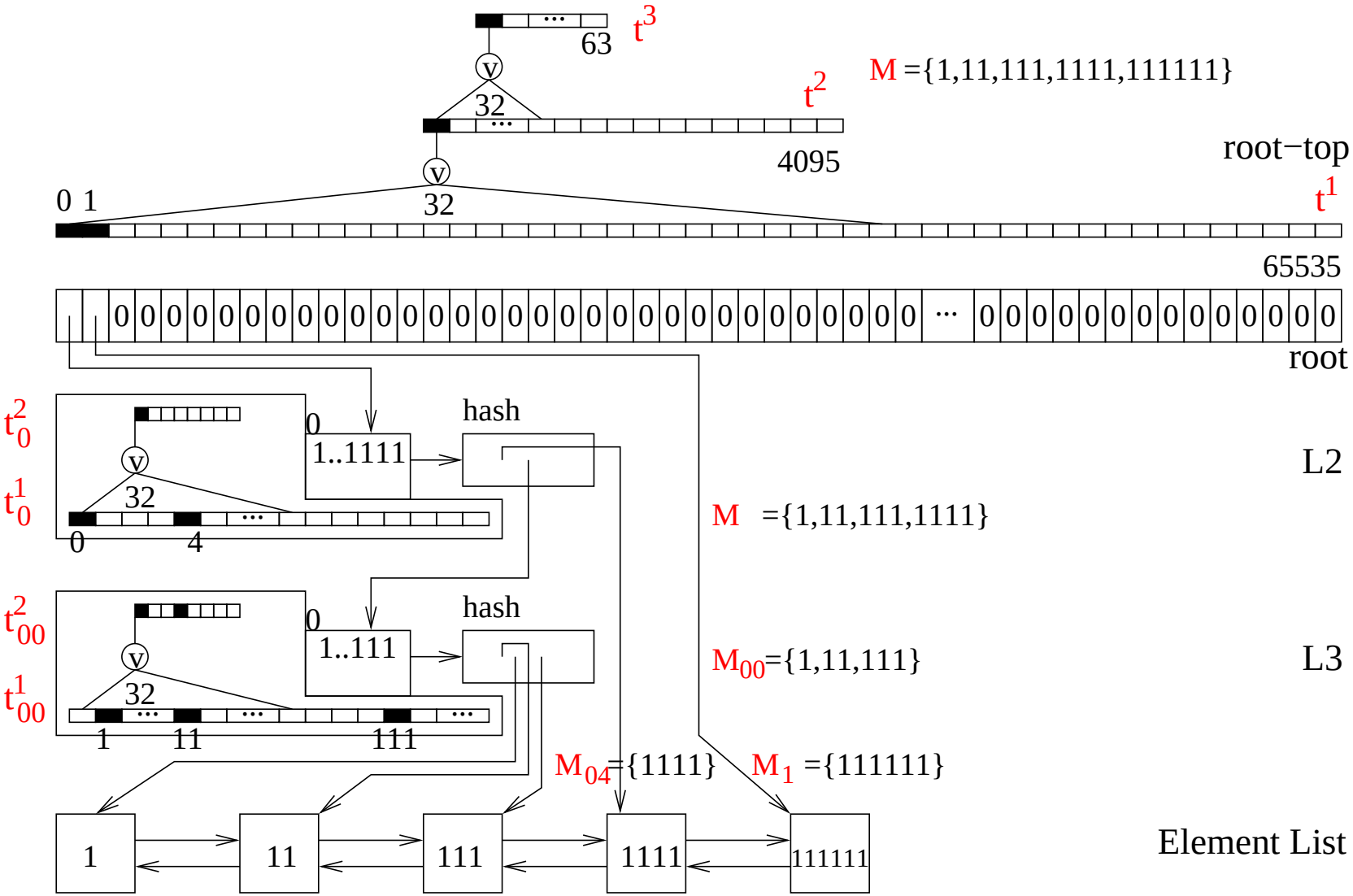


Efficient 32 bit Implementation

Sorted doubly linked lists for associated information and range queries



Example


63
t^3
v
32
t^2
4095
$M = \{1,11,111,1111,111111\}$
root−top
0 1
32
t^1
65535
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ··· 0 0 0 0 0 0 0 0 0 0 0 0 0 0
root
t_0^2
t_0^1
0
1..1111
hash
L2
$M = \{1,11,111,1111\}$
0
4
t_{00}^2
t_{00}^1
1..111
$M_{00}=\{1,11,111\}$
L3
1
11
111
$M_{04}=\{1111\}$
$M_1 = \{111111\}$
1111
111111
Element List

Locate High Level

// return handle of $\min x \in M : y \leq x$

Function locate$(y : \mathbb{N})$: ElementHandle

if $y > \max M$ **then return** ∞

$i := y[16..31]$ // Level 1

if $r[i] = \text{nil} \vee y > \max M_i$ **then return** $\min M_{t^1}$.locate(i)

if $M_i = \{x\}$ **then return** x

$j := y[8..15]$ // Level 2

if $r_i[j] = \text{nil} \vee y > \max M_{ij}$ **then return** $\min M_{i,t_i^1}$.locate(j)

if $M_{ij} = \{x\}$ **then return** x

return $r_{ij}[t_{ij}^1$.locate$(y[0..7])]$ // Level 3

Locate in Bit Arrays

// find the smallest $j \geq i$ such that $t^k[j] = 1$

Method locate(i) for a bit array t^k consisting of n bit words

// $n = 32$ for $t^1, t^2, t^1_i, t^1_{ij}$; $n = 64$ for t^3; $n = 8$ for t^2_i, t^2_{ij}

assert some bit in t^k to the right of i is nonzero

j := i **div** n // which word?

$a := t^k[nj..nj+n-1]$

set $a[(i \bmod n)+1..n-1]$ to zero // $n-1 \cdots i \bmod n \cdots 0$

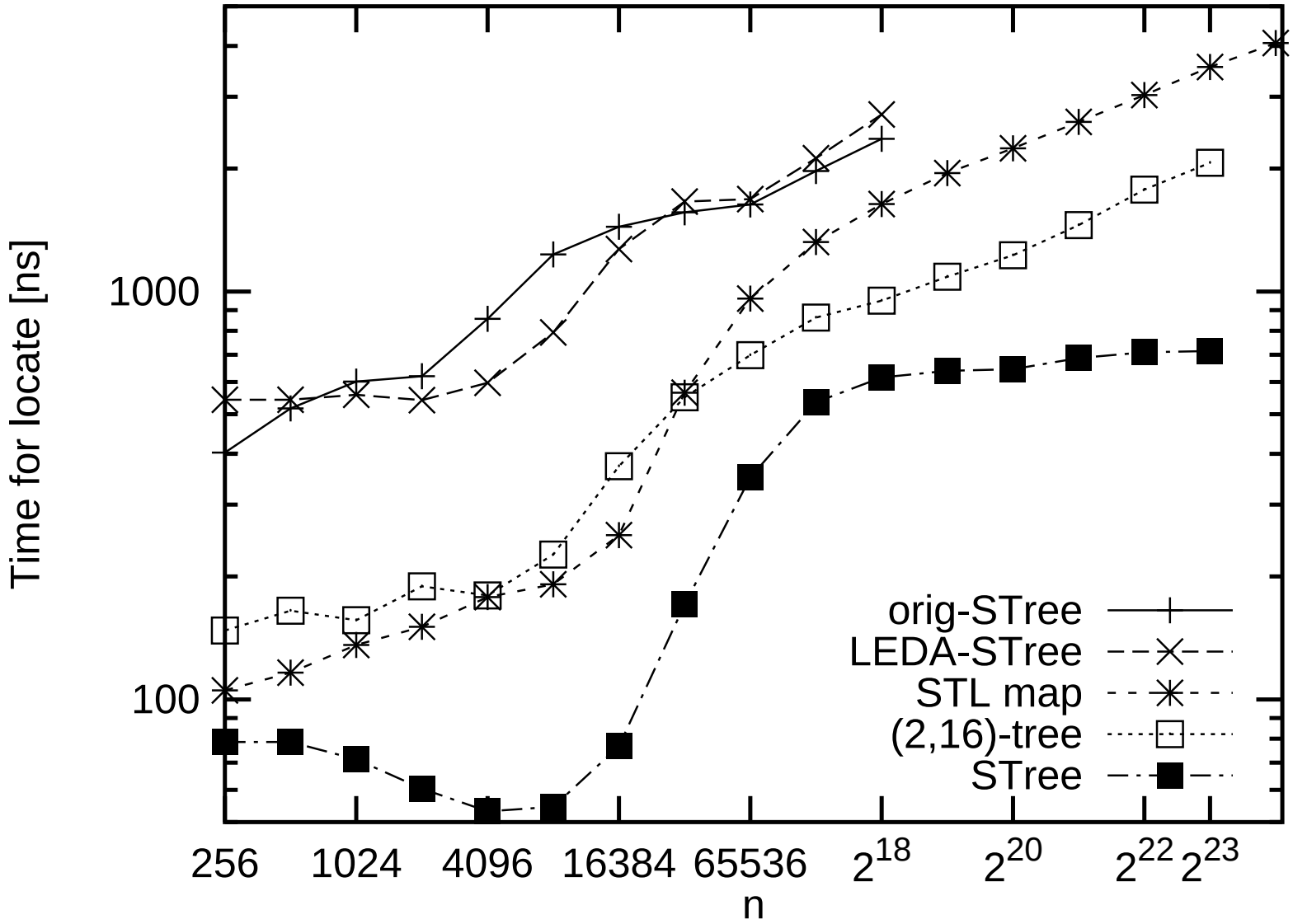
if $a = 0$ **then**

$j := t^{k+1}$.locate(j)

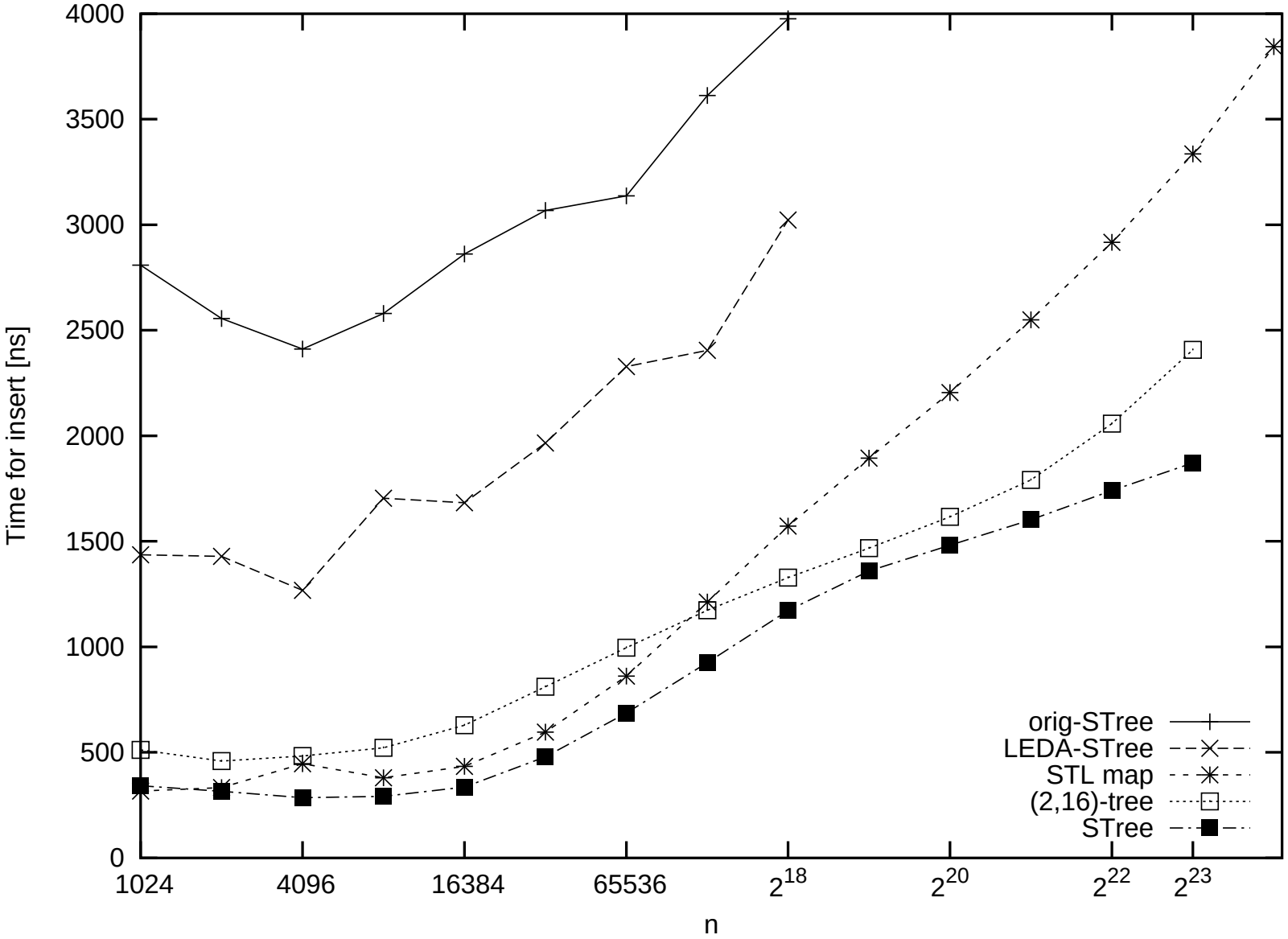
$a := t^k[nj..nj+n-1]$

return $nj +$ msbPos(a) // e.g. floating point conversion

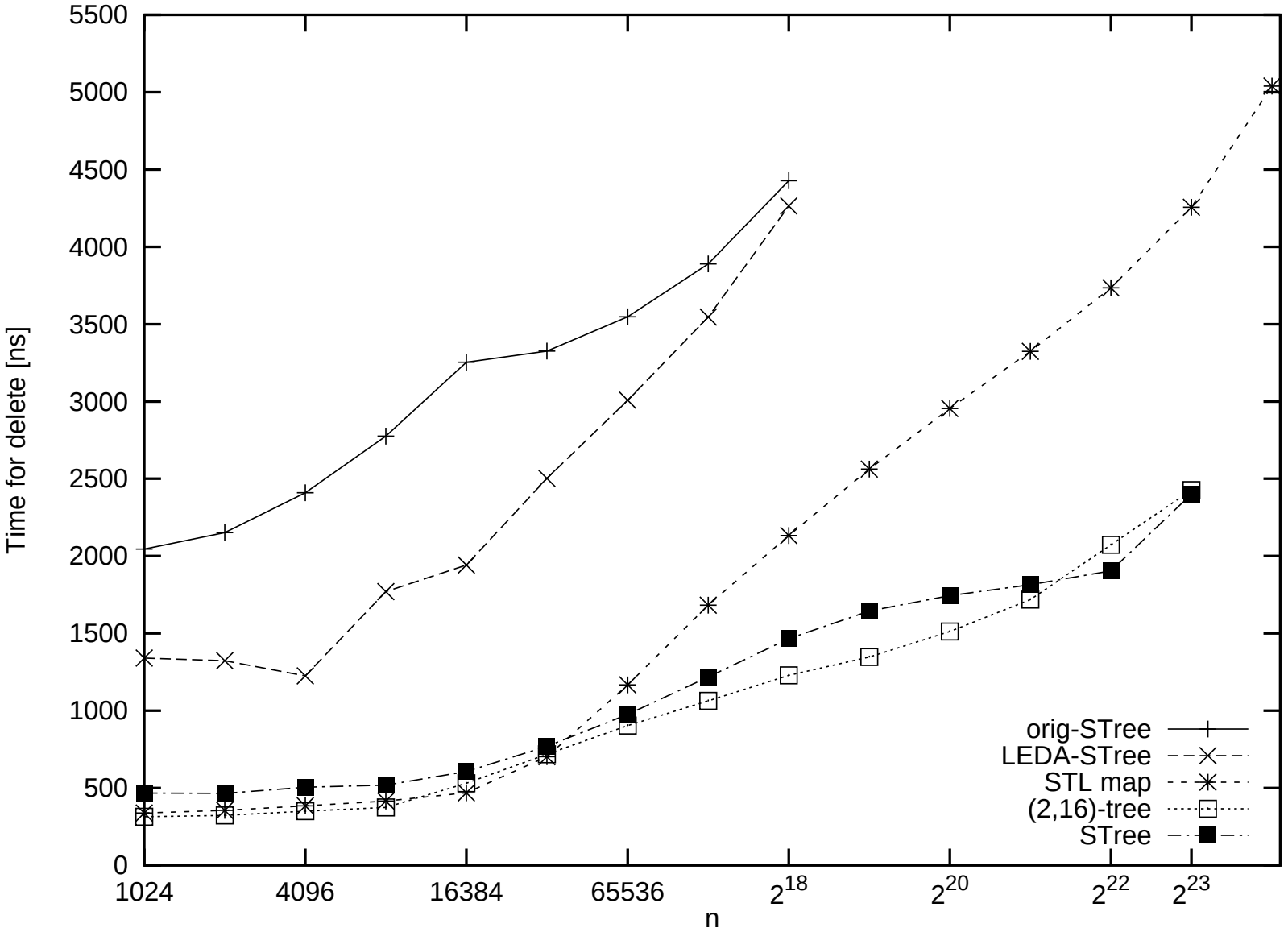
Random Locate



Random Insert



Delete Random Elements



Open Problems

- Measurement for “worst case” inputs
- Measure Performance for realistic inputs
  - IP lookup etc.
  - Best first heuristics like, e.g., bin packing
- More space efficient implementation
- (A few) more bits

3 Hashing

"to hash" $\approx$ "to bring into complete disorder"

paradoxically, this helps us to find things more easily!

store set $M \subseteq$ Element.

key(e) is unique for $e \in M$.

support dictionary operations in $O(1)$ time:

M.insert$(e :$ Element$)$: $M := M \cup \{e\}$

M.remove$(k :$ Key$)$: $M := M \setminus \{e\}, e = k$

M.find$(k :$ Key$)$: return $e \in M$ with $e = k$; $\perp$ if none present

(Convention: key is *implicit*), e.g. $e = k$ iff key$(e) = k$)

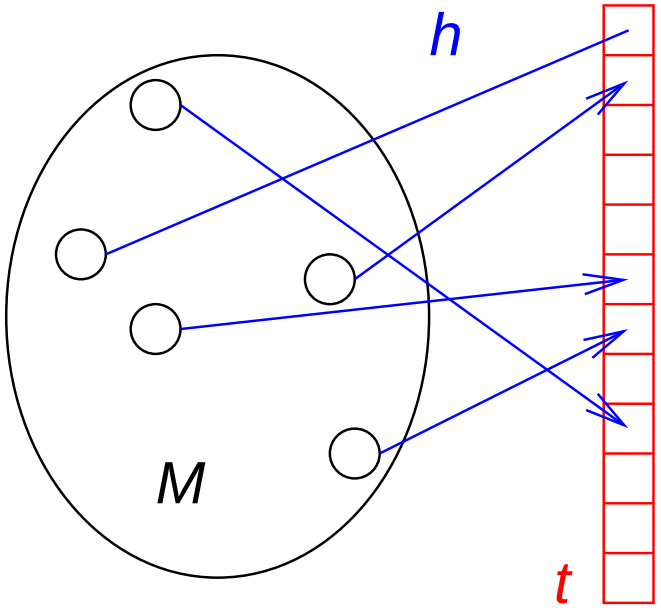
An (Over)optimistic approach

A (perfect) hash function h

maps elements of M to

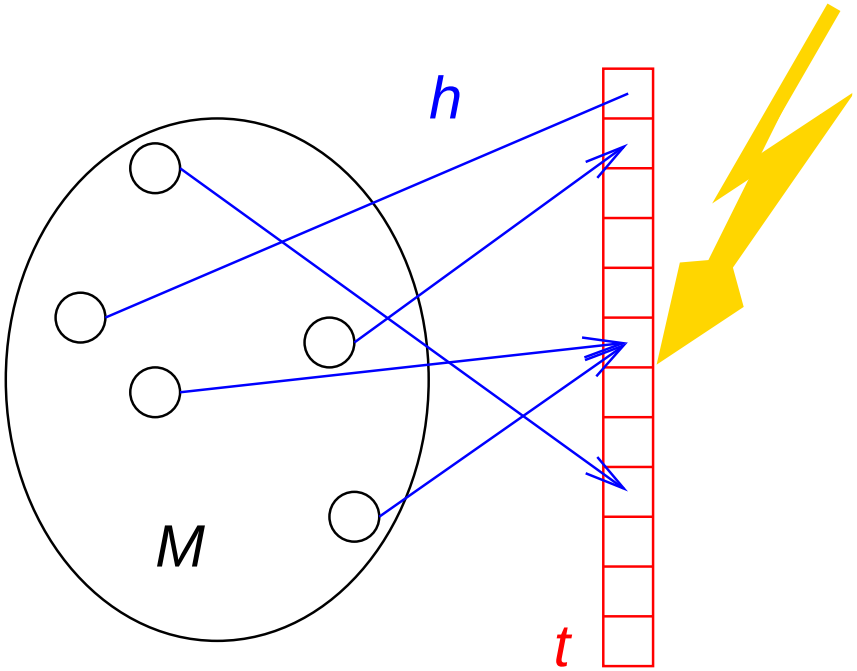
unique entries of table $t[0..m-1]$, i.e.,

$t[h(\text{key}(e))] = e$



Collisions

perfect hash functions are difficult to obtain

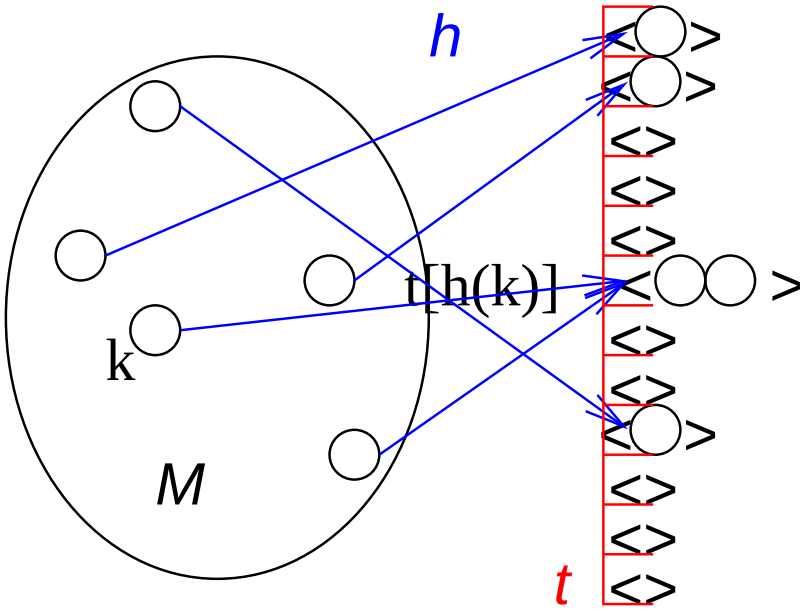


Example: Birthday Paradoxon

Collision Resolution

for example by closed hashing

entries: elements $\rightsquigarrow$ sequences of elements



Hashing with Chaining

Implement sequences in closed hashing by singly linked lists

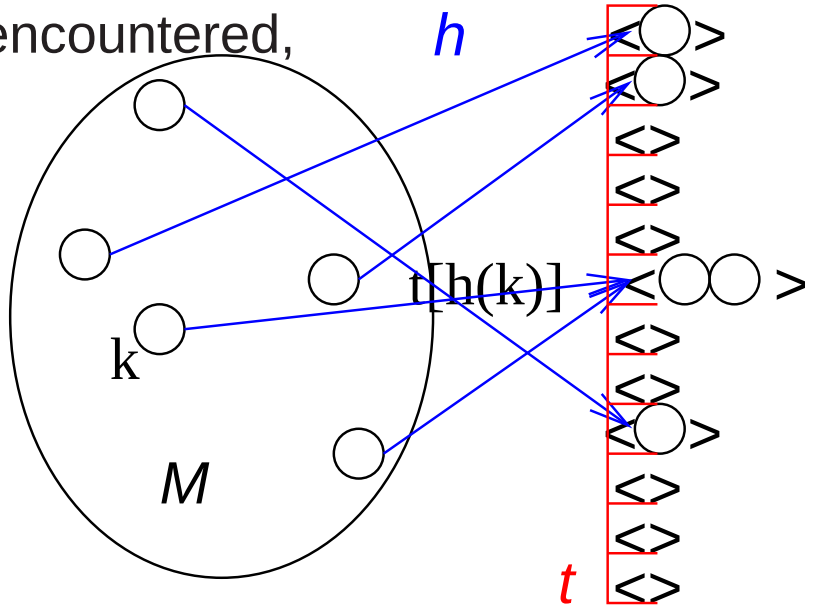
insert(e): Insert e at the beginning of $t[h(e)]$. constant time

remove(k): Scan through $t[h(k)]$. If an element e with $h(e) = k$ is encountered, remove it and return.

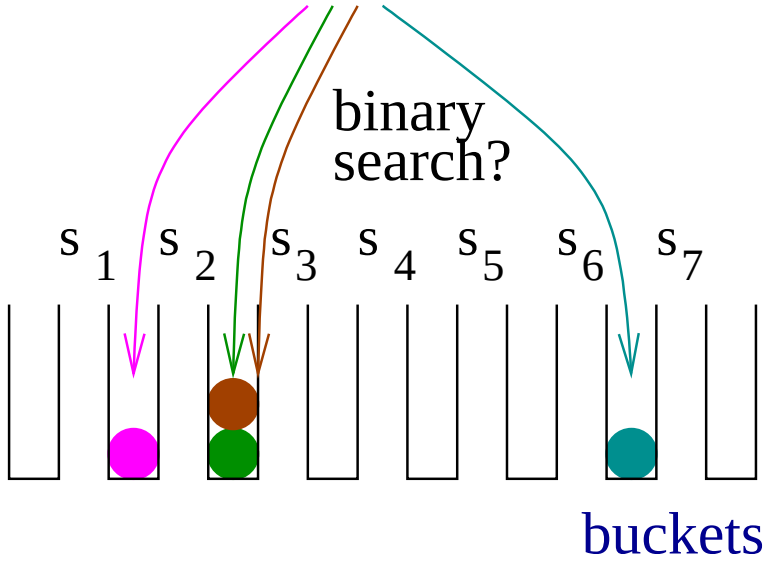
find(k) : Scan through $t[h(k)]$.

If an element e with $h(e) = k$ is encountered, return it. Otherwise, return $\perp$.

$O(|M|)$ worst case time for remove and find



A Review of Probability



sample space Ω

events: subsets of Ω

p_x =probability of $x \in \Omega$

$\mathbb{P}[\mathcal{E}] = \sum_{x \in E} p_x$

random variable $X_0 : \Omega \to \mathbb{R}$

Example from hashing

random hash functions $\{0..m-1\}^{\text{Key}}$

$\mathcal{E}_{42} = \{h \in \Omega : h(4) = h(2)\}$

uniform distr. $p_h = m^{-|\text{Key}|}$

$\mathbb{P}[\mathcal{E}_{42}] = \frac{1}{m}$

$X = |\{e \in M : h(e) = 0\}|.$

expectation $\mathrm{E}[X_0] = \sum_{y \in \Omega} p_y X(y)$ $\mathrm{E}[X] = \frac{|M|}{m}$ (*)

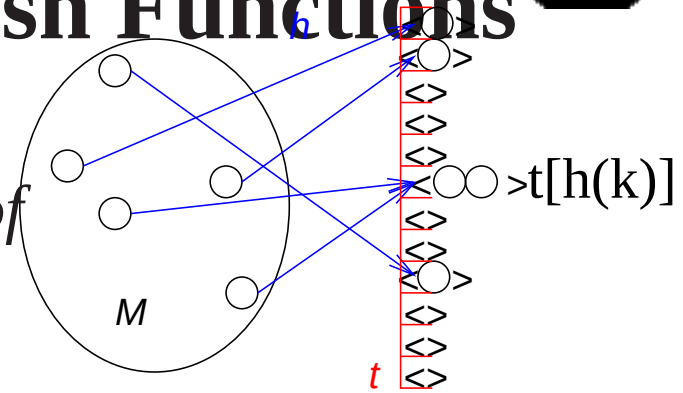
Linearity of Expectation: $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$

Proof of (*):

Consider the 0-1 RV $X_e = 1$ if $h(e) = 0$ for $e \in M$ and $X_e = 0$ else.

$$\mathrm{E}[X_0] = \mathrm{E}[\sum_{e \in M} X_e] = \sum_{e \in M} \mathrm{E}[X_e] = \sum_{e \in M} \mathbb{P}[X_e = 1] = |M| \cdot \frac{1}{m}$$

Analysis for Random Hash Functions



Satz 1. *The expected execution time of* remove(k) *and* find(k) *is* $O(1)$ *if* $|M| = O(m)$.

Proof. Constant time plus the time for scanning $t[h(k)]$.

$X := |t[h(k)]| = |\{e \in M : h(e) = h(k)\}|$.

Consider the 0-1 RV $X_e = 1$ if $h(e) = h(k)$ for $e \in M$ and $X_e = 0$ else.

$$\mathrm{E}[X] = \mathrm{E}[\sum_{e \in M} X_e] = \sum_{e \in M} \mathrm{E}[X_e] = \sum_{e \in M} \mathbb{P}[X_e = 1] = \frac{|M|}{m}$$
$$= O(1)$$

This is independent of the input set M. □

Universal Hashing

Idea: use only certain "easy" hash functions

Definition:

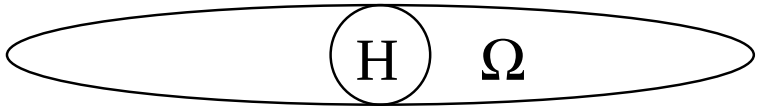
$\mathcal{U} \subseteq \{0..m-1\}^{\text{Key}}$ is *universal*

if for all x, y in Key with $x \neq y$ and random $h \in \mathcal{U}$,

$$\mathbb{P}[h(x) = h(y)] = \frac{1}{m} \ .$$

Satz 2. *Theorem 1 also applies to universal families of hash functions.*

Proof. For $\Omega = \mathcal{U}$ we still have $\mathbb{P}[X_e = 1] = \frac{1}{m}$.
The rest is as before. □



H
Ω

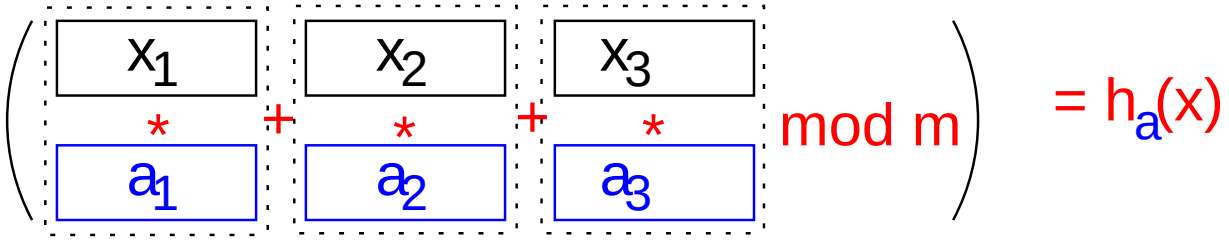
A Simple Universal Family

Assume m is prime, Key $\subseteq \{0,\ldots,m-1\}^k$

Satz 3. *For* $\mathbf{a} = (a_1,\ldots,a_k) \in \{0,\ldots,m-1\}^k$ *define*

$h_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x} \bmod m$, $H^{\cdot} = \left\{h_{\mathbf{a}} : \mathbf{a} \in \{0..m-1\}^k\right\}$.

$H^{\cdot}$ *is a universal family of hash functions*



Proof. Consider $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_k)$ with $x_j \neq y_j$
count **a**-s with $h_{\mathbf{a}}(\mathbf{x}) = h_{\mathbf{a}}(\mathbf{y})$.
For each choice of a_is, $i \neq j$, $\exists$ exactly one a_j with
$h_{\mathbf{a}}(\mathbf{x}) = h_{\mathbf{a}}(\mathbf{y})$:

$$\sum_{1 \leq i \leq k} a_i x_i \equiv \sum_{1 \leq i \leq k} a_i y_i \; (\mathbf{mod}\; m)$$

$$\Leftrightarrow a_j(x_j - y_j) \equiv \sum_{i \neq j, 1 \leq i \leq k} a_i(y_i - x_i) \; (\mathbf{mod}\; m)$$

$$\Leftrightarrow a_j \equiv (x_j - y_j)^{-1} \sum_{i \neq j, 1 \leq i \leq k} a_i(y_i - x_i) \; (\mathbf{mod}\; m)$$

m^{k-1} ways to choose the a_i with $i \neq j$.
m^k is total number of **a**s, i.e.,

$$\mathbb{P}\left[h_{\mathbf{a}}(x) = h_{\mathbf{a}}(\mathbf{y})\right] = \frac{m^{k-1}}{m^k} = \frac{1}{m}.$$

□

Bit Based Universal Families

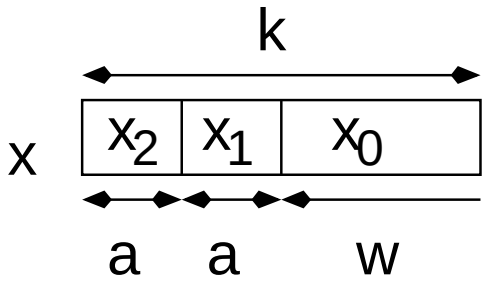
Let $m = 2^w$, Key $= \{0,1\}^k$

Bit-Matrix Multiplication: $H^{\oplus} = \left\{ h_{\mathbf{M}} : \mathbf{M} \in \{0,1\}^{w \times k} \right\}$

where $h_{\mathbf{M}}(\mathbf{x}) = \mathbf{M}x$ (arithmetics $\bmod 2$, i.e., xor, and)

Table Lookup: $H^{\oplus[]} = \left\{ h^{\oplus[]}_{(t_1,\ldots,t_b)} : t_i \in \{0..m-1\}^{\{0..w-1\}} \right\}$

where $h^{\oplus[]}_{(t_1,\ldots,t_b)}((x_0,x_1,\ldots,x_b)) = x_0 \oplus \bigoplus_{i=1}^{b} t_i[x_i]$



Hashing with Linear Probing

Open hashing: go back to original idea.

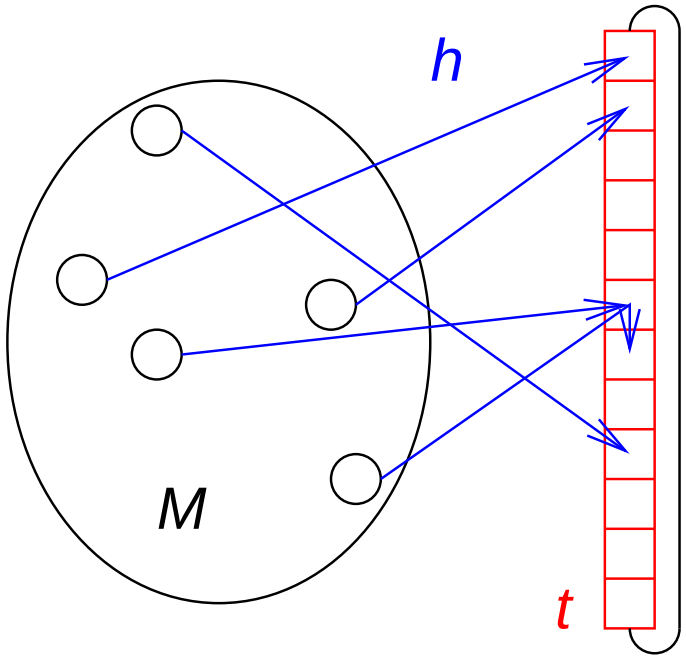
Elements are directly stored in the table.

Collisions are resolved by finding other entries.

linear probing: search for next free place by scanning the table.

Wrap around at the end.

- simple
- space efficient
- cache efficient



The Easy Part

Class BoundedLinearProbing$(m, m' : \mathbb{N}; h : \text{Key} \to 0..m-1)$

$t=[\bot,\ldots,\bot]$: **Array** $[0..m+m'-1]$ **of** Element

invariant $\forall i : t[i] \neq \bot : \forall j \in \{h(t[i])..i-1\} : t[i] \neq \bot$

Procedure insert$(e : \text{Element})$

for $i := h(e)$ **to** ∞ **while** $t[i] \neq \bot$ **do** ;

assert $i < m+m'-1$

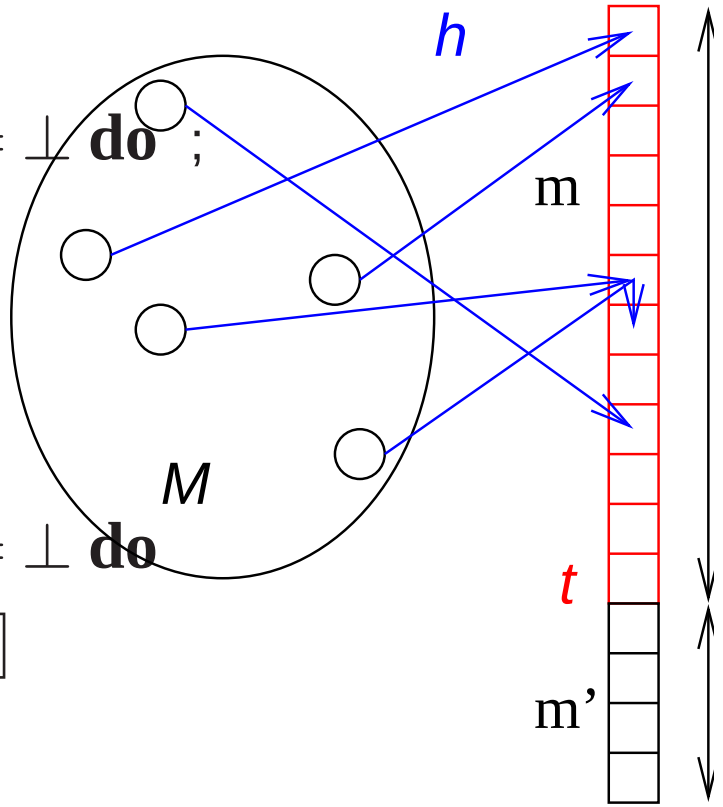
$t[i] := e$

Function find$(k : \text{Key}) : \text{Element}$

for $i := h(e)$ **to** ∞ **while** $t[i] \neq \bot$ **do**

if $t[i] = k$ **then return** $t[i]$

return $\bot$



Remove

example: $t = [\ldots, \underset{h(z)}{x}, y, z, \ldots]$, remove$(x)$

invariant $\forall i : t[i] \neq \bot : \forall j \in \{h(t[i])..i-1\} : t[i] \neq \bot$

Procedure remove$(k : \text{Key})$

```
for i := h(k) to ∞ while k ≠ t[i] do                    // search k
    if t[i] = ⊥ then return                              // nothing to do
//we plan for a hole at i.
for j := i+1 to ∞ while t[j] ≠ ⊥ do
    //Establish invariant for t[j].
    if h(t[j]) ≤ i then
        t[i] := t[j]                                     // Overwrite removed element
        i := j                                           // move planned hole
t[i] := ⊥                                                // erase freed entry
```

More Hashing Issues

- High probability and worst case guarantees

  $\rightsquigarrow$ more requirements on the hash functions

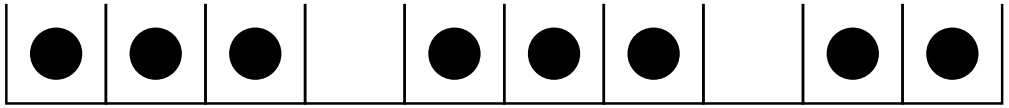
- Hashing as a means of load balancing in parallel systems, e.g., storage servers
  - Different disk sizes and speeds
  - Adding disks / replacing failed disks without much copying

Space Efficient Hashing with Worst Case Constant Access Time

Represent a set of n elements (with associated information) using space $(1+\varepsilon)n$.

Support operations insert, delete, lookup, (doall) efficiently.

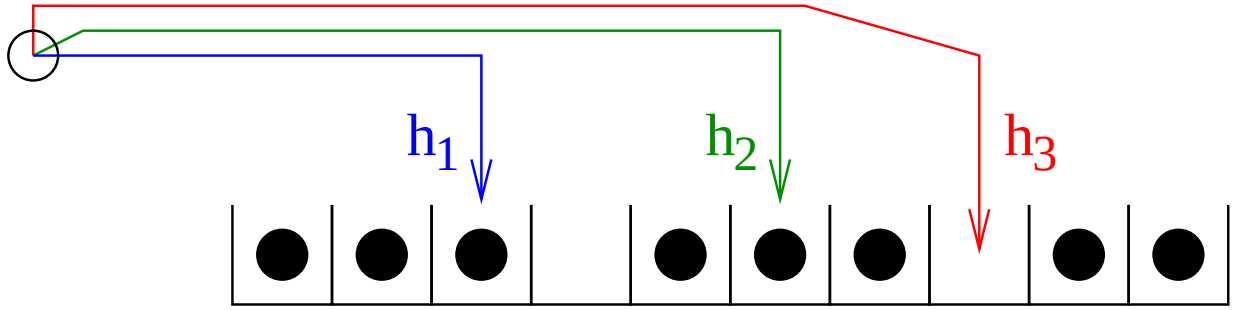
Assume a truly random hash function h



Related Work

Uniform hashing:

Expected time $\approx \frac{1}{\varepsilon}$

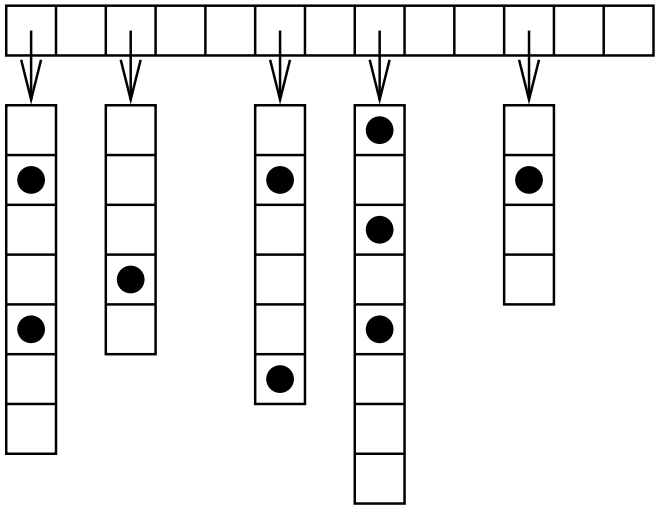


Dynamic Perfect Hashing,

[Dietzfelbinger et al. 94]

Worst case constant time

for lookup but ε is not small.



Approaching the Information Theoretic Lower Bound:

[Brodnik Munro 99,Raman Rao 02]

Space $(1+o(1))\times$lower bound without associated information

[Pagh 01] static case.

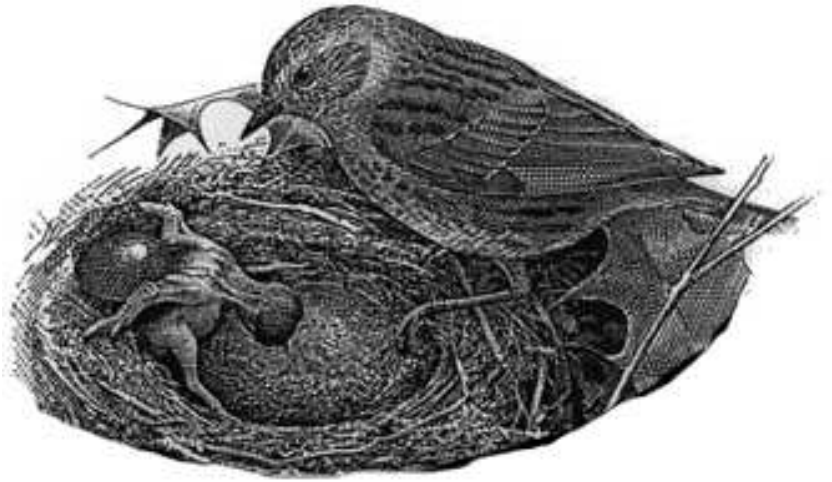
Cuckoo Hashing

[Pagh Rodler 01] Table of size $2+\varepsilon$

Two choices for each element.

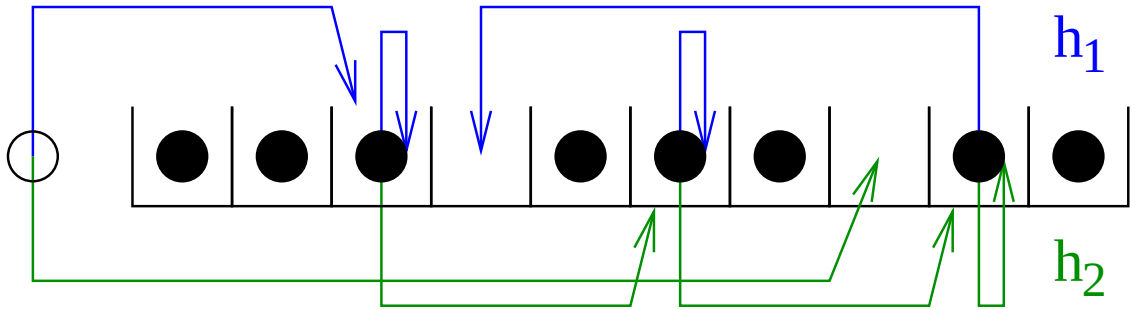
Insert moves elements;

rebuild if necessary.



Very fast lookup and insert.

Expected constant insertion time.



d-ary Cuckoo Hashing

d choices for each element.

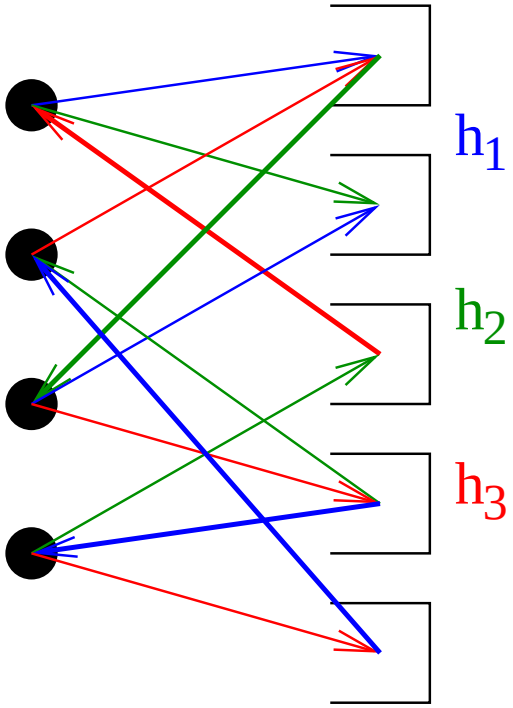
Worst case d probes for delete and lookup.

Task: maintain perfect matching

in the bipartite graph

$(L = \text{Elements}, R = \text{Cells}, E = \text{Choices})$,

e.g., insert by BFS of random walk.

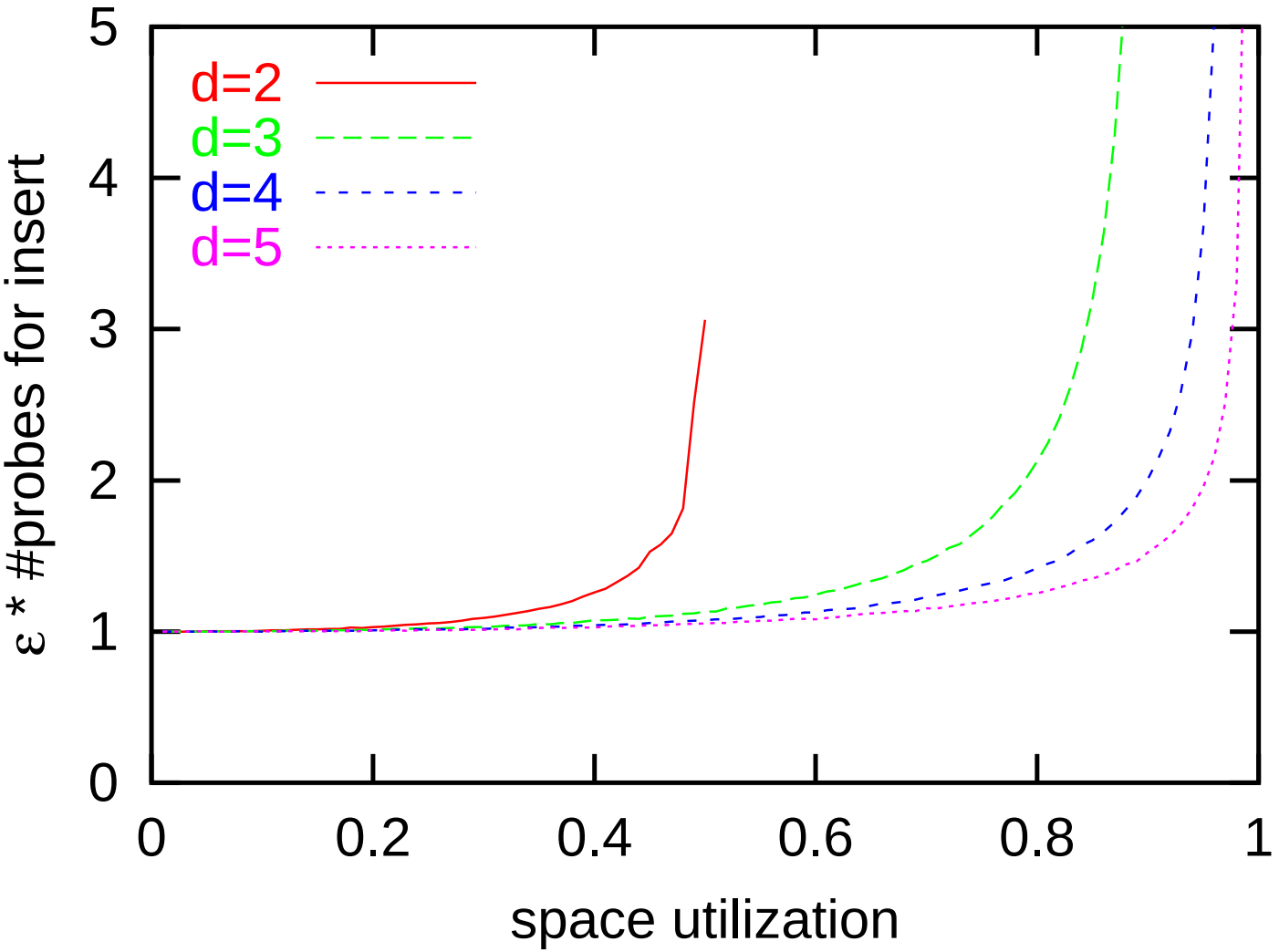


Tradeoff: Space $\leftrightarrow$ Lookup/Deletion Time

Lookup and Delete: $d = O\left(\log \frac{1}{\varepsilon}\right)$ probes

Insert: $\left(\frac{1}{\varepsilon}\right)^{O(\log(1/\varepsilon))}$, (experiments) $\longrightarrow O(1/\varepsilon)$?

Experiments



Open Questions and Further Results

- Tight analysis of insertion
- Two choices with d slots each [Dietzfelbinger et al.]

  $\rightsquigarrow$ cache efficiency

Good Implementation?

- Automatic rehash
- Always correct