On the Development and Use of Differential Analyzers

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Abstract

In this essay, we present the development of an important analogue calculating device, the *differential analyzer*. Section 1 introduces the main purpose that this type of machine was built for, namely the solution of differential equations and, in this context, the integration of a function. Section 2 summarizes the most important milestones regarding the development of mechanical differential analyzers. Section 3 deals with some typical applications. Section 4 concludes with some improvements of the differential analyzer that were achieved by replacing mechanical parts with electrical ones. Furthermore, a short comparison of the differential analyzer with the present-day technology is done.

1 Introduction

An ordinary differential equation of m-th order can be written in explicit form as $y^{(m)}(x) = f(x, y(x), y'(x), y''(x), \ldots, y^{(m-2)}(x), y^{(m-1)}(x))$ [Hei00, p. 153]. The function f is given, while the function y has to be determined. To get started, let us deal with a very simple example, namely $m = 2$ and $f(x, y(x), y'(x)) = y(x)$. Thus, we have the following second-order differential equation: $y''(x) = y(x)$. An equivalent notation of this equation is $y(x) = \int\int y(x)\, dx\, dx$. At this point, the principle difficulty can be observed that arises when a differential equation has to be solved: y appears on both sides of the equation, i.e., the value of the function y at the point x depends on the integral of y. Furthermore, we can realize that integrals can play a distinct role with respect to the solution of a differential equation. We will pick up on these points soon, but first we want to introduce one physical example in order to demonstrate that differential equations are not just a "toy" of mathematicians, but have direct reference to real world applications.

If a capacitor and a resistor appear in a series connection, the total voltage U sums up to $U(t) = Q(t)/C + R \cdot I(t)$. Furthermore, the charge Q of the capacitor can be expressed by $Q(t) = \int I(t)dt$. Hence, we obtain $U(t) = 1/C \int I(t)dt + R \cdot I(t) \iff I(t) = 1/R \left(U(t) - 1/C \int I(t)dt\right)$. Thus, we have exactly the same situation as in the simple theoretical example. The current I at the point of time t depends on the current I because I contributes to the integrand. A similar example with an inductance and a resistor is presented in [Har49, p. 4].

The following basic strategy, which forms the foundation of a differential analyzer, can be used to solve such differential equations.

1. Take an initial value $y_0 = y(0) = a$.
2. Evaluate the right side of the equation, i.e., the integral, for a "small" Δx and obtain a value y_1.
3. Use the obtained y_1 as input and re-evaluate the right side.
4. Iterate.

The correct result is obtained for $\Delta x \to 0$, i.e., Δx has to become an infinitesimal small dx.

Obviously, the main component of this strategy is the integrator (also often called *planimeter* [Hor14, p. 190]), and the next section will present how such an integrator has been implemented and what other parts are required for a differential analyzer, a machine that can solve differential equations.

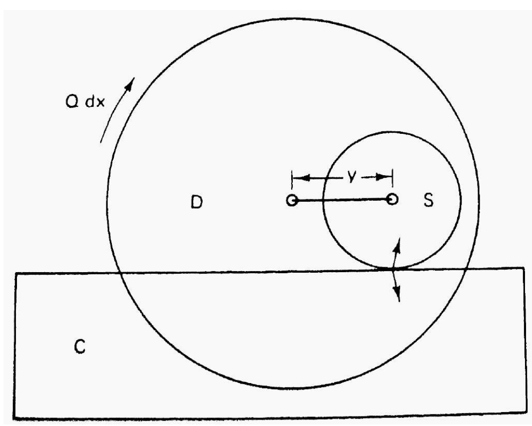
2 Development of the Differential Analyzer

2.1 Basics

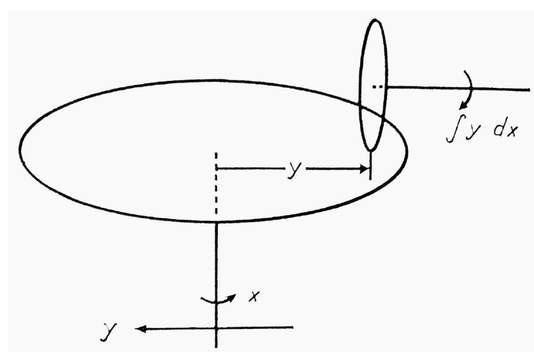
The first integrators and differential analyzers consisted only of mechanical components. In these machines, a variable is represented by a shaft and the value of the variable by the rotation of the shaft. In order to build an integrator that computes $y = \int f(x)\,dx$, we need some kind of gear between a driving shaft x and a driven shaft y, where the ratio of the gear is specified by $f(x)$. When the shaft x is rotated by an infinitesimal small amount dx, the shaft y is rotated by $f(x)\,dx$, and when we continue the rotation of x, the rotation of y sums up to $\int f(x)\,dx$. The prerequisite for this method is the possibility of selecting continuously the appropriate gear ratio. [Har49, p. 5]

According to [Hor14, p. 190], the first attempts of implementing such a device took probably place in 1814 and in the following years many other attempts followed, but the results were either not adequately published or the machines were too inaccurate to be reliable. The breakthrough was achieved by James Thomson, the brother of Lord Kelvin [Wil97, p. 201], published in 1876 [Hor14, p. 192]. His design (see Figure 1(a)) consists of a disk that is rotated by a driving shaft, a cylinder that is connected with the driven shaft that represents the value of the integral, and a sphere that is the "mediator" between the disk and the sphere. The sphere has always contact with both the disk and the cylinder, but it can be placed to different positions on a line between the center of the disk and the border. The farer the sphere is away from the center, the greater is the gear ratio.

A simpler integrator can be built using two disks that are arranged perpendicular (see Figure 1(b)), one is connected with the driving shaft x, the other with the driven shaft y. Again, the point of contact can be arbitrarily selected on a line between the center of the driving disk and its border [Har49, p. 5].



(a) The disk-sphere-cylinder integrator



(b) The disk-wheel integrator

Figure 1: Two types of integrators [Wil97, p. 202], [Har49, p. 5]

2.2 Bush's Differential Analyzer at the MIT

While Lord Kelvin in 1876 already thought about the combination of several of his brother's integrators in order to solve differential equations [Har49, p. 7], it took 55 years until a first differential analyzer could be realized. In 1931, Vannevar Bush constructed a working machine at the MIT [Wil97, p. 203]. One major problem that Bush was able to master was the slip of the mechanical parts that interacted only by friction. When several components of this kind are combined, it is likely that the tension gets so big that the friction between two wheels is not sufficient so that they slip; this, of course, leads to a falsification of the results. The way out that was successfully implemented by Bush is the installation of torque amplifiers for shafts that are used simultaneously as output and input, i.e., a driven shaft that is also a driving shaft (for the next component) is broken and a torque amplifier is installed inbetween. The amplifier takes the driven shaft as input and rotates the outgoing driving shaft exactly the same way, but by more power.

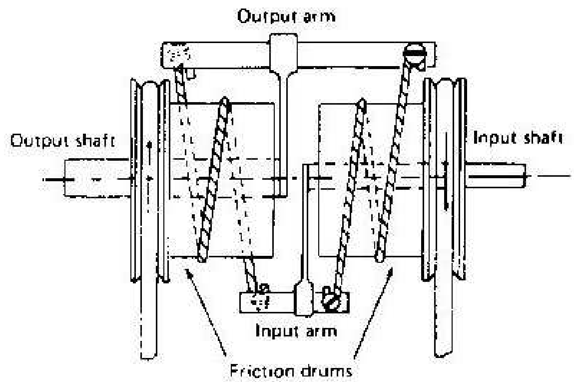


Figure 2: Principle of Bush's torque amplifier (from the 1931 article by Bush, taken from [Wil00])

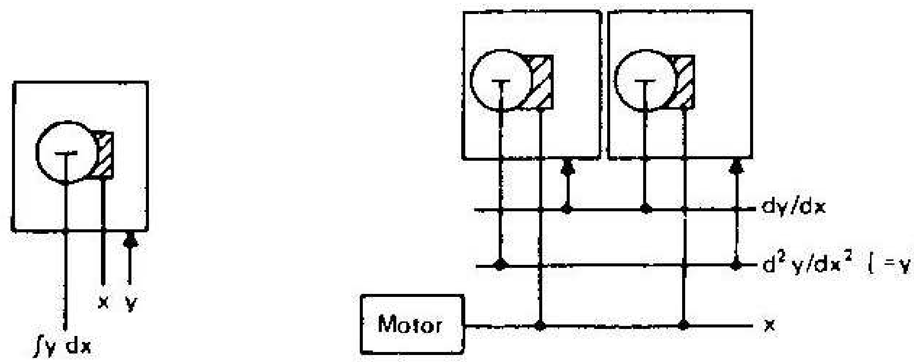
To achieve this aim, Bush took advantage of the principle of the ship's capstan in a quite inventive way. Figure 2 demonstrates his design. Both the input and the output shaft are connected with an arm each. Around the shafts is a friction drum each. The frictions drums are continuously rotated by a powerful motor. The input and output arms are connected by two threads that are winded round one of the friction drums each. The rolling direction of the threads and of the friction drums is important: looking at the amplifier on Figure 2 from the left, on the output side, the friction drum rotates counter-clockwise and the cord from the input to the output arm is winded clockwise, while on the input side, the friction drum rotates clockwise and the thread is rolled counter-clockwise. The invariant of this system is that the input and the output arms are always opposite to each other. The threads are so long that they are just loosely rolled round the drums when the arms are exactly opposite to each other. When the input shaft is rotated in either direction, exactly one of both threads is tensed. In order to tense a thread, only a minor amount of power is required. Now, a friction between that thread and the corresponding friction drum arises. Hence, the cord is pulled in the rolling direction of the drum – that is driven by the powerful motor. Therefore, the output arm is pulled with much power. Due to the above mentioned choice of the rolling directions of the friction drums and of the cords, it is ensured that the output arm is always dragged in the correct direction, namely in the same as the input arm. As soon as the output shaft – and consequently the output arm – approaches the correct position, the thread is loosed and the friction decreases so that the output shaft does not overshoot the mark. [Wil00]

Beside the integrators and amplifiers, Bush's differential analyzer consisted of gears for constant multiplications and gears for doing addition and subtraction [Wil97, p. 204].

Figure 3(b) demonstrates how a differential analyzer could be used to solve the introductory example $y''(x) = y(x)$. The used schematic notation for an integrator is given in Figure 3(a). For the sake of simplicity, the amplifiers are omitted. The right integrator integrates y (resp. y'') once and outputs $\int y\,dx$ (resp. y'). The left integrator integrates the output of the right one and outputs $\int\int y\,dx\,dx$ (resp. the "new" y).

2.3 The Developments in the United Kingdom

Since Bush's differential analyzer was able to solve a wide range of differential equations and since such equations appeared in many applications – while it was usually difficult to solve them without a machine –, there were many people that copied Bush's machine. A common problem



(a) Schematic notation for an integrator

(b) Schematic notation for the differential analyzer

Figure 3: Setup for solving the equation $y''(x) = y(x)$ [Wil00]

for some of these imitators was to convince their financial backers of the usefulness of a differential analyzer. An inventive solution for this problem was found by Douglas Hartree who built a model of a differential analyzer together with A. Porter at the University of Manchester in 1934 [Har49, p. 13]. The costs could be kept very low because almost every part of the machine was built using Meccano, a kind of construction set that was a popular toy for boys at this time. Only for the integrator disks, ground glass instead of a Meccano part was used, and some other parts, particularly for the amplifiers, had to be built using other materials [Irw02]. The Meccano model was surprisingly successful and its accuracy was in the order of 2% so that it could be used for serious applications [Har49, p. 14]. Due to this success, one year later a full scale machine was built by Metropolitan Vickers.

A similar way was chosen at the University of Cambridge at the instance of J. E. Lennard-Jones. In 1935, J. B. Bratt built a model of a differential analyzer. Similar to Hartree's machine, Meccano was used for most parts apart from those that were crucial to the precision of the machine [Wil85, p. 25]. In 1939, a full scaled machine replaced the Meccano model.

Of course, the development and the use of differential analyzers – inspired both by Bush's original work and by the British Meccano models – was not limited to the USA and the UK, but spread out throughout Europe (e.g., Germany and Norway) and North America. For instance, one Meccano machine was installed by Beatrice "Trixie" Worsley at the University of Toronto in the early 1950s [Wil97, p. 205].

3 Applications

The development of the differential analyzer was advanced mainly by people who were interested in the applications of the machine rather than in the machine itself from the engineering point of view. Of course, the people who built those machines had to deal with mechanical problems that arose and it can be observed that these problems were often dominant in comparison with the theoretical problems that usually had been solved much earlier. For example, as already mentioned, it took 55 years from Lord Kelvin's theoretical deliberations regarding a differential analyzer to Bush's first realization. However, the applications were always the mainspring.

Lord Kelvin applied the planimeter developed by his brother to his *tide calculating machine* (which is no actual differential analyzer, but shares one of its main components) [Hor14, p. 193]. Such a machine was quite important since it was able to predict the tide for a given time in the future with sufficient accuracy. This information was essential for ships that called at a harbour in order to judge if the tide was high enough to reach the harbour safely without touching some rocks on the ground [Wil97, p. 198].

Vannevar Bush dealt with differential equations related to the electric power network. He started to solve the equations analytically, but he soon realized that this took far too much time so that he decided to concentrate his efforts on the construction of a machine that would take this time-consuming task over [Wil97, p. 204]. Similarly, Douglas Hartree was no engineer but a physicist and, particularly, an expert in numerical methods of computation [Wil85, p. 107]. At the University of Cambridge, Maurice V. Wilkes used the differential analyzer for different applications. He investigated the propagation of long-wave radio waves, supported E. Monroe in solving a differential equation emerging in the two-centre problem in wave machines, and analysed a model of graphite with respect to the interrelationship between the potential energy and the situation of the carbon atoms [Wil85, pp. 25–27].

In [Har49, p. 25] some other applications are listed that demonstrate the wide range of use. The examples belong to the fields of

- physics (e.g. "motion of electrified particles in the magnetic field of the earth"),
- electronics (e.g. "problems in non-linear electrical circuits"),
- chemistry (e.g. "chemical kinetics"), and
- scheduling (e.g. "running times of railroad trains").

However, the most frequent field of application, at least during the Second World War, was probably a military one – in fact across all borders, i.e., in the USA, in the UK and in Germany. One common task was the computation of ballistic firing tables. For instance, with the help of the MIT, the British tried to calculate the ballistic trajectory of the German V2 rockets [Wil97, p. 206].

4 Further Developments and the State of the Art

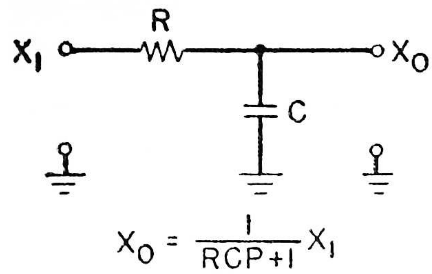
4.1 Further Developments

While the first differential analyzers consisted exclusively of mechanical parts, during the further development, more and more parts were replaced by electrical ones so that as intermediate step electromechanical machines and finally purely electrical differential analyzers were built. The basic concepts have never changed, i.e., on principle, the electrical parts that replaced the mechanical ones fulfilled the same functionality as their predecessors.

Again, Vannevar Bush counts to the precursors. In 1945, he built together with S. H. Caldwell a new differential analyzer at the MIT. While he still used mechanical integrators, he replaced the mechanical connections between the shafts by electrical ones. The azimuth of the output shaft about its axis is encoded by a pair of variable condensers into an electrical value. Similarly, the azimuth of the input shaft that should be driven by the above mentioned output shaft is represented by an electrical value. Both values are compared in order to calculate the difference, which should be minimized. This comparison is done by a unit that controls a motor that drives the input shaft so that both shafts are synchronized [Har49, p. 14].

Later on, differential analyzers were constructed where the remaining mechanical parts were replaced by electrical ones. For instance, an electrical integrator can be built using a condenser (see Figure 4).

The advantages of the electrical differential analyzer over the mechanical one are manifold. First of all, the computation can be done much faster due to the high speed that electrical components operate at. Furthermore, it is more convenient to setup the machine to an initial condition as a control desk can be used for this purpose. In contrast, "setting up a mechanical differential analyzer was not a job for anyone who liked to keep his hands clean" [Wil85, p. 30], because of the fact that you had to deal directly with the oily shafts and gears.



P is short for $\frac{d}{d\tau}$ [KK56, p. 9].

Figure 4: An electrical integrator [KK56, p. 13]

4.2 State of the Art

Although the mechanical differential analyzers had already been very useful and electrical ones were even much more sophisticated, this type of machine practically died out – because it was an analogue one. In [Wil85, p. 123], the computer pioneer M. V. Wilkes describes his reaction when he was asked in 1946 if there was a future for the differential analyzer: "This was not a question that I had consciously considered, but I found myself saying no, and from that moment on I had no doubt in my mind that the days of analogue devices for scientific computation were numbered." But why ? The advantages of digital machines had become visible. Nowadays, differential equations are solved by numerical algorithms on a digital computer. On principle, these algorithms often base on the same strategy as the differential analyzers, which is described in Section 1. However, there is one essential difference. The numerical algorithms on digital computers always deal with discrete steps [Hei00, p. 160], never with infinitesimal small ones as a differential analyzer does this. Hence, at first sight, the analogue differential analyzer is *better* than a modern digital computer because *in theory* it computes the exact result, while a digital computer only approximates the solution. But, *in practice*, an analogue machine can never operate with 100% accuracy, i.e., it deviates from the exact solution and the crucial disadvantage is the fact that it is barely possible to set a bound for the deviation. Worse is the fact that it is generally not even possible to *get* a bound for the deviation, i.e., you do not know how good the results are that you have obtained. Of course, you can realize that *usually* the deviation is less than 1%, but if you get the results of just one computation, you cannot exclude that this time the deviation is bigger because of some missing oil, for example.

The numerical algorithms on digital computers have both properties. Firstly, for a given algorithm and the selected parameters (particularly, the selected step size), you can compute an error bound that is *never* exceeded. Secondly, you can choose the parameters of the algorithm in order to set the error bound. In order to get better results, you have to reduce the step size – in general, you do not have to build a new machine in order to improve the accuracy of the results.

List of Figures

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